A new derivation of Jeffery's equation

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Abstract

In this article, we present a modern derivation of Jeffery's equation for the motion of a small rigid body immersed in a Navier-Stokes flow, using methods of asymptotic analysis. While Jeffery's result represents the leading order equations of a singularly perturbed flow problem involving ellipsoidal bodies, our formulation is for bodies of general shape and we also derive the equations of the next relevant order.

Keywords. Jeffery's equation, small immersed rigid body, asymptotic analysis
AMS subject classifications.

1 Introduction

The original work of Jeffery [9], contains a derivation of approximate equations of motion for a rigid ellipsoidal body immersed in a surrounding linear flow field. While few details are given about the simplifying assumptions underlying the derivation, the main focus is put on a technical integral representation of Stokes solutions around general ellipsoids.
Jeffery's equations are widely used in the theory of suspensions where one tries to discover how the motion of a suspended particle and the suspending liquid influence each other. The single particle dynamics is then a basic ingredient for statistical approaches which model the behavior of (dilute) ensembles of suspended particles. An extension of Jeffery's work to more general geometries can be found, for example, in [1] and [2] (for additional results on the topic, we refer to the review article [14].) An important application of the theory is the description of injection molding of fiber reinforced plastics [15]. Moreover, the approach parallels in several respects the Leslie-Ericksen theory of rigid-rod liquid-crystalline polymers in the nematic phase [3, 11] and is also used in connection with electrorheological fluids [6].
In view of the importance of Jeffery's equation, we think it is worth while revisiting the derivation. In contrast to Jeffery's approach we are going to stress the basic assumptions in the derivation and try to keep our considerations largely independent of the particular ellipsoidal geometry. Our argument is based on an asymptotic expansion in ε (the size of the body) of the fluid velocity

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and pressure fields $\boldsymbol{u}, p$ as well as the center of mass coordinates and the angular velocity $\boldsymbol{c}, \boldsymbol{\omega}$ of the rigid body (see figure 1) which satisfy a system of differential equations.

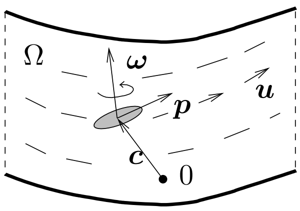

Figure 1: A small rigid body immersed in a flowing liquid.

It turns out that the leading order velocity field $\boldsymbol{u}_0$ is the undisturbed flow (i.e. without the rigid body) and that the center of mass of the body, also in leading order, follows the streamlines

$$\dot{\boldsymbol{c}}_0(t) = \boldsymbol{u}_0(t, \boldsymbol{c}_0(t)). \tag{1}$$

In the case of an elongated, rotationally symmetric ellipsoid, the orientation vector $\boldsymbol{p}$ in direction of the major semi-axis (see figure 1) obeys Jeffery's equation

$$\dot{\boldsymbol{p}}_0 = \frac{1}{2}(\operatorname{curl} \boldsymbol{u}_0) \wedge \boldsymbol{p}_0 + \lambda(S[\boldsymbol{u}_0]\boldsymbol{p}_0 - (\boldsymbol{p}_0^T S[\boldsymbol{u}_0]\boldsymbol{p}_0)\boldsymbol{p}_0) \tag{2}$$

up to an error of order ε, Here $S[\boldsymbol{u}_0]$ is the symmetric part of the velocity Jacobian, $\wedge$ denotes the cross product, and the parameter

$$\lambda = \frac{(l/d)^2 - 1}{(l/d)^2 + 1}$$

is a function of the ratio l/d between the length of the long and the short semi-axis of the ellipsoidal body.

Equation (2) appears as solvability condition of a Stokes problem which describes the first order local flow around the small body (an explicit form of which is derived in Jeffery's article for the case of the ellipsoid). Such Stokes problems are generally known as *mobility problems* and have been carefully studied (see, for example, [10]). In our approach, we extend these results by deriving also the equations for the *second order* coefficients and we show *quantitatively* how well the approximate solution obtained from the truncated asymptotic expansion satisfies the coupled system of Navier-Stokes and rigid body equations. The latter result is a first step towards a mathematical proof that Jeffery's equation is the correct asymptotic description of the ellipsoid dynamics.

Our derivation of (2) is organized as follows. In section 2, we introduce the relevant equations which describe the moving particle. A non-dimensionalization leads to an ε-dependent system of differential equations where ε is the ratio of body size versus the typical size of the flow domain. In section 3, we motivate our ansatz for the asymptotic expansion and present the result for the two leading orders (details of the computation are given in appendix B). The solvability condition for the equations defining the expansion coefficients then give rise to ordinary differential equations for the rotation state of the body and the first order perturbation of the center of mass. The geometrical information about the body enters through weighted surface averages of six solutions of the Stokes equation in the exterior domain. For general geometries, these values can be calculated using, for example, a boundary element method. However, in the special case of ellipsoidal geometries, the exact values are known, leading to Jeffery's equation if the ellipsoid has rotational symmetry (see section 4 for details). Finally, in section 5, some particular solutions of Jeffery's equation are presented to illustrate the approximate motion of ellipsoidal bodies in surrounding flows.

2 Equations of motion

Our aim is to describe the motion of a small rigid body εE in a flowing liquid. We assume that the body template $E \subset \mathbb{R}^3$ is an open and bounded domain with a smooth surface ∂E, constant density ρ_b and center of mass at the origin, i.e.

$$\int_E \boldsymbol{y}\, d\boldsymbol{y} = 0. \tag{3}$$

Our main example will be the rotationally symmetric ellipsoid

$$E = \left\{ \boldsymbol{y} \in \mathbb{R}^3 \, : \, \frac{y_1^2}{l^2} + \frac{y_2^2 + y_3^2}{d^2} < 1 \right\}, \qquad l, d > 0 \tag{4}$$

whose orientation in space can be described by a vector $\boldsymbol{p}$ which points in the direction of the rotation axis. The forces which drive the rigid body originate in fluid friction and pressure acting on its surface. Since the body is moving, we find a Navier-Stokes problem with moving boundary for the flow variables. To describe location and velocity of this boundary, we first consider the kinematics of the rigid body. In the next step, we formulate the flow equations and, finally, we specify the rigid body dynamics.

2.1 Rigid body kinematics

The motion of a rigid body consists of translations and rotations. In particular, the position and orientation of the rigid body $E_\varepsilon(t)$ at time t is completely characterized by its center of mass $\boldsymbol{c}(t) \in \mathbb{R}^3$ and a rotation matrix $R(t) \in SO(3)$

$$E_\varepsilon(t) = \varepsilon R(t) E + \boldsymbol{c}(t), \qquad t \geq 0.$$

If we trace a body point $\boldsymbol{y} \in E$, it follows the path $\boldsymbol{x}(t) = \varepsilon R(t)\boldsymbol{y} + \boldsymbol{c}(t)$. Its velocity consists of the linear velocity $\dot{\boldsymbol{c}}(t)$ and the angular velocity $\varepsilon \dot{R}(t)\boldsymbol{y}$ to be investigated further. Writing $\dot{R}(t)$ as

$$\dot{R}(t) = \left(\lim_{h \to 0} \frac{R(t+h)R^T(t) - I}{h} \right) R(t), \tag{5}$$

we are led to the rotation matrix $R(t+h)R^T(t)$ which is, for small h, a slight perturbation of the identity I. Its derivative at $h = 0$ is a so called *infinitesimal rotation* (an element of the tangent space to $SO(3)$ at the point I). An elementary calculation (see for example [4, 5] or appendix A) shows that the infinitesimal rotations in $\mathbb{R}^3$ are the skew symmetric matrices which can be written as

$$B(\boldsymbol{\omega}) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega} \in \mathbb{R}^3. \tag{6}$$

We remark that $B(\boldsymbol{\omega})\boldsymbol{x}$ is exactly the vector product between $\boldsymbol{\omega}$ and $\boldsymbol{x}$ which we denote with $\wedge$, i.e.

$$B(\boldsymbol{\omega})\boldsymbol{x} = \boldsymbol{\omega} \wedge \boldsymbol{x}, \qquad \boldsymbol{\omega}, \boldsymbol{x} \in \mathbb{R}^3. \tag{7}$$

Relation (5) can now be cast into the form

$$\dot{R}(t) = B(\boldsymbol{\omega}(t))R(t) \tag{8}$$

with a suitable vector $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ called angular velocity. In section 2.3, we will describe the dynamics of the rigid body by specifying equations for $\boldsymbol{c}$ and $\boldsymbol{\omega}$. Note that the rotation matrix $R(t)$ controlling the position of the rigid body may be recovered from $\boldsymbol{\omega}$ by solving (8) with some initial value $R(0)$.
Coming back to the velocity of a body point $\boldsymbol{x}(t) = \varepsilon R(t)\boldsymbol{y} + \boldsymbol{c}(t)$, we find

$$\varepsilon \dot{R}(t)\boldsymbol{y} = \varepsilon B(\boldsymbol{\omega}(t))R(t)\boldsymbol{y} = B(\boldsymbol{\omega}(t))(\boldsymbol{x}(t) - \boldsymbol{c}(t))$$

leading to the velocity equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{\omega}(t) \wedge (\boldsymbol{x}(t) - \boldsymbol{c}(t)) + \dot{\boldsymbol{c}}(t). \tag{9}$$

From (9) we infer the evolution equation for an orientation vector $\boldsymbol{p}(t)$ pointing from the center of mass $\boldsymbol{c}(t)$ to some body point $\boldsymbol{x}(t)$

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{\omega}(t) \wedge \boldsymbol{p}(t). \tag{10}$$

In case of the ellipsoid (4) we will later focus on the orientation vector $\boldsymbol{e}_1$ which points along the major semi-axis.

2.2 The flow problem

We assume that both the liquid and the rigid body $E_\varepsilon(t)$ are contained in a regular domain $\Omega \subset \mathbb{R}^3$. The liquid should be incompressible with constant density ρ_f and kinematic viscosity ν. Its pressure p and velocity field $\boldsymbol{u}$ satisfy the Navier-Stokes equation in $\Omega \backslash E_\varepsilon(t)$

$$\text{div}\boldsymbol{u} = 0, \qquad \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p/\rho_f = \nu \Delta \boldsymbol{u} \tag{11}$$

complemented by suitable initial and boundary values. Without specifying details, we assume that the boundary values on $\partial\Omega$ are chosen in such a way that the undisturbed flow problem (no immersed body) is well posed. If a rigid body $E_\varepsilon(t) = \varepsilon R(t)E + \boldsymbol{c}(t)$ is present in Ω, the no-slip condition on its boundary implies in view of (9)

$$\boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{\omega}(t) \wedge (\boldsymbol{x} - \boldsymbol{c}(t)) + \dot{\boldsymbol{c}}(t), \qquad \boldsymbol{x} \in \partial E_\varepsilon(t). \tag{12}$$

To non-dimensionalize the equations, we choose a length scale L and a velocity scale U which are typical for the undisturbed flow. The corresponding time scale is $\tau = L/U$. As scale for viscous stress and pressure, we select $\Sigma = \rho_f \nu U/L$. Then, the scaled functions $\hat{\boldsymbol{u}}(\hat{t}, \hat{\boldsymbol{x}}) = \boldsymbol{u}(\tau\hat{t}, L\hat{\boldsymbol{x}})/U$ and $\hat{p}(\hat{t}, \hat{\boldsymbol{x}}) = p(\tau\hat{t}, L\hat{\boldsymbol{x}})/\Sigma$ satisfy

$$\mathrm{div}_{\hat{\boldsymbol{x}}} \hat{\boldsymbol{u}} = 0, \qquad \mathrm{Re}(\partial_{\hat{t}} \hat{\boldsymbol{u}} + \hat{\boldsymbol{u}} \cdot \nabla_{\hat{\boldsymbol{x}}} \hat{\boldsymbol{u}}) = \mathrm{div}_{\hat{\boldsymbol{x}}} \hat{\sigma}$$

where Re is the Reynolds number and $\hat{\sigma}$ the stress tensor

$$\mathrm{Re} = \frac{UL}{\nu}, \qquad \hat{\sigma} = -\hat{p}I + 2S[\hat{\boldsymbol{u}}], \qquad S[\hat{\boldsymbol{u}}] = (\nabla_{\hat{\boldsymbol{x}}} \hat{\boldsymbol{u}} + \nabla_{\hat{\boldsymbol{x}}} \hat{\boldsymbol{u}}^T)/2.$$

On the rigid body surface, we find

$$\hat{\boldsymbol{u}}(\hat{t}, \hat{\boldsymbol{x}}) = \hat{\boldsymbol{\omega}}(\hat{t}) \wedge (\hat{\boldsymbol{x}} - \hat{\boldsymbol{c}}(\hat{t})) + \frac{d}{d\hat{t}} \hat{\boldsymbol{c}}(\hat{t}), \qquad \hat{\boldsymbol{x}} \in \hat{E}_{\hat{\varepsilon}}(\hat{t})$$

where $\hat{\boldsymbol{c}}(\hat{t}) = \boldsymbol{c}(\tau\hat{t})/L$, $\hat{\boldsymbol{\omega}}(\hat{t}) = \tau\boldsymbol{\omega}(\tau\hat{t})$, and

$$\hat{E}_{\hat{\varepsilon}}(\hat{t}) = \hat{\varepsilon}\hat{R}(\hat{t})E + \hat{\boldsymbol{c}}(\hat{t}), \qquad \hat{\varepsilon} = \varepsilon/L, \quad \hat{R}(\hat{t}) = R(\tau\hat{t}).$$

2.3 Rigid body dynamics

In section 2.1 we have derived the velocity field inside the rigid body as

$$\boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{\omega}(t) \wedge (\boldsymbol{x} - \boldsymbol{c}(t)) + \dot{\boldsymbol{c}}(t), \qquad \boldsymbol{x} \in E_\varepsilon(t)$$

so that the total linear momentum is given by

$$\int_{E_\varepsilon(t)} \rho_b \boldsymbol{u}(t, \boldsymbol{x})\, d\boldsymbol{x} = \rho_b |E_\varepsilon(t)| \dot{\boldsymbol{c}}(t) = \rho_b \varepsilon^3 |E| \dot{\boldsymbol{c}}(t). \tag{13}$$

Here we have used (3) and $|\cdot|$ to denote the volume of a set. According to Newton's law, the rate of change of (13) balances the forces acting on the body which, in the present case, originate from the fluid stress σ acting on the boundary

$$\rho_b \varepsilon^3 |E| \ddot{\boldsymbol{c}}(t) = \int_{\partial E_\varepsilon(t)} \sigma \boldsymbol{n}\, dS. \tag{14}$$

Here, $\boldsymbol{n}$ is the normal field pointing out of $E_\varepsilon(t)$. For the total angular momentum with respect to the center of mass, we find

$$\boldsymbol{L}(t) = \int_{E_\varepsilon(t)} \rho_b (\boldsymbol{x} - \boldsymbol{c}(t)) \wedge (\boldsymbol{u}(t, \boldsymbol{x}) - \boldsymbol{c}(t))\, d\boldsymbol{x}$$
$$= -\rho_b \int_{E_\varepsilon(t)} (\boldsymbol{x} - \boldsymbol{c}(t)) \wedge \big[(\boldsymbol{x} - \boldsymbol{c}(t)) \wedge \boldsymbol{\omega}(t)\big]\, d\boldsymbol{x}.$$

Using relation (7) we can write the vector products with $\boldsymbol{x} - \boldsymbol{c}$ in terms of the skew symmetric matrix $B(\boldsymbol{x} - \boldsymbol{c})$ which eventually leads to

$$\boldsymbol{L}(t) = \rho_b \varepsilon^5 \int_E -B^2(R(t)\boldsymbol{y})\, d\boldsymbol{y}\, \boldsymbol{\omega}(t).$$

Finally, with relations (53) and (55) from the appendix, we get

$$\boldsymbol{L}(t) = \rho_b \varepsilon^5 |E| T(t) \boldsymbol{\omega}(t)$$

where $T(t)$ is the inertia tensor

$$T(t) = R(t) T R^T(t), \qquad T = \frac{1}{|E|} \int_E |\boldsymbol{y}|^2 I - \boldsymbol{y} \otimes \boldsymbol{y} \, d\boldsymbol{y}. \tag{15}$$

The equation for angular momentum now implies that the rate of change of $\boldsymbol{L}$ is balanced by the angular momentum generated from the fluid stress on the surface

$$\rho_b \varepsilon^5 |E| \frac{d}{dt} \left(T(t) \boldsymbol{\omega}(t) \right) = \int_{\partial E_\varepsilon(t)} (\boldsymbol{x} - \boldsymbol{c}(t)) \wedge \sigma \boldsymbol{n}(\boldsymbol{x}) \, dS. \tag{16}$$

Introducing the same scaling as in the previous section, we derive from (14) and (16)

$$\begin{aligned} \hat{\varepsilon} \varrho \mathrm{Re} \left(\frac{d}{d\hat{t}} \right)^2 \hat{\boldsymbol{c}}(\hat{t}) &= \frac{1}{|E|} \int_{\partial E} \hat{\sigma}(\hat{t}, \hat{\boldsymbol{X}}(\hat{t}, \boldsymbol{y}, \hat{\varepsilon})) \hat{R}(\hat{t}) \hat{\boldsymbol{n}}(\boldsymbol{y}) \, dS \\ \hat{\varepsilon}^2 \varrho \mathrm{Re} \frac{d}{d\hat{t}} \left(\hat{T}(\hat{t}) \hat{\boldsymbol{\omega}}(\hat{t}) \right) &= \frac{1}{|E|} \int_{\partial E} (\hat{R}(\hat{t}) \boldsymbol{y}) \wedge \hat{\sigma}(\hat{t}, \hat{\boldsymbol{X}}(\hat{t}, \boldsymbol{y}, \hat{\varepsilon})) \hat{R}(\hat{t}) \boldsymbol{n}(\boldsymbol{y}) \, dS \end{aligned}$$

where $\hat{\boldsymbol{n}}$ is the outer normal field on ∂E, $\hat{T}(\hat{t}) = \hat{R}(\hat{t}) T \hat{R}^T(\hat{t})$ and

$$\hat{\boldsymbol{X}}(\hat{t}, \boldsymbol{y}, \hat{\varepsilon}) = \hat{\varepsilon} \hat{R}(\hat{t}) \boldsymbol{y} + \hat{\boldsymbol{c}}(\hat{t}).$$

The factor ϱ is the quotient of body and fluid density $\varrho = \rho_b / \rho_f$.

2.4 Summary

The unknowns in our problem are the flow variables $p, \boldsymbol{u}$ and the rigid body parameters $\boldsymbol{c}, R$. Since we continue to work with the scaled variables, the hat superscripts are dropped from now on. With the mapping

$$\boldsymbol{X}(t, \boldsymbol{y}, \varepsilon) = \varepsilon R(t) \boldsymbol{y} + \boldsymbol{c}(t), \qquad \boldsymbol{y} \in \mathbb{R}^3$$

we can write the set $E_\varepsilon(t) = \boldsymbol{X}(t, E, \varepsilon)$ occupied by the rigid body at time t in terms of the template body E which has its center of mass at $\boldsymbol{y} = 0$. The inverse mapping

$$\boldsymbol{Y}(t, \boldsymbol{x}, \varepsilon) = R^T(t) \frac{\boldsymbol{x} - \boldsymbol{c}(t)}{\varepsilon}, \qquad \boldsymbol{x} \in \mathbb{R}^3 \tag{17}$$

yields the template coordinates $\boldsymbol{y}$ corresponding to a space point $\boldsymbol{x}$.
On $\Omega_\varepsilon(t) = \mathrm{int}(\Omega \backslash E_\varepsilon(t))$, the variables $\boldsymbol{u}, p$ satisfy the Navier-Stokes equation

$$\mathrm{div} \boldsymbol{u} = 0, \qquad \mathrm{Re}(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) = -\nabla p + \Delta \boldsymbol{u} \tag{18}$$

with boundary conditions on $\partial \Omega$, initial conditions, and

$$\boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{\omega}(t) \wedge (\boldsymbol{x} - \boldsymbol{c}(t)) + \dot{\boldsymbol{c}}(t), \qquad \boldsymbol{x} \in \partial E_\varepsilon(t). \tag{19}$$

Further, we have

$$\varepsilon\varrho\mathrm{Re}\,\ddot{\boldsymbol{c}}(t) = \frac{1}{|E|}\int_{\partial E}\sigma(t,\boldsymbol{X}(t,\boldsymbol{y},\varepsilon))R(t)\boldsymbol{n}(\boldsymbol{y})\,dS \tag{20}$$

$$\varepsilon^2\varrho\mathrm{Re}\frac{d}{dt}\left(T(t)\boldsymbol{\omega}(t)\right) = \frac{1}{|E|}R(t)\int_{\partial E}\boldsymbol{y}\wedge R^T(t)\sigma(t,\boldsymbol{X}(t,\boldsymbol{y},\varepsilon))R(t)\boldsymbol{n}(\boldsymbol{y})\,dS \tag{21}$$

and

$$\dot{R}(t) = B(\boldsymbol{\omega}(t))R(t) \tag{22}$$

complemented with suitable initial conditions. Here $\sigma_{ij} = -p\delta_{ij} + 2S_{ij}[\boldsymbol{u}]$ is the fluid stress tensor with $S_{ij}[\boldsymbol{u}] = (\partial_{x_j}u_i + \partial_{x_i}u_j)/2$ and $T(t)$ is the inertia tensor

$$T(t) = R(t)TR^T(t), \qquad T = \frac{1}{|E|}\int_E |\boldsymbol{y}|^2 I - \boldsymbol{y}\otimes\boldsymbol{y}\,d\boldsymbol{y}. \tag{23}$$

We remark that (21) and (22) constitute a non-linear, second order differential equation for R because $\boldsymbol{\omega}$ can easily be calculated from (22) as

$$\omega_1 = (\dot{R}R^T)_{32}, \qquad \omega_2 = (\dot{R}R^T)_{13}, \qquad \omega_3 = (\dot{R}R^T)_{21}.$$

Nevertheless, we keep $\boldsymbol{\omega}$ as variable to simplify notation and because of the physical relevance.

3 Asymptotic expansion

3.1 Basic assumptions

In order to obtain Jeffery's equations (1), (2) as leading order dynamics of the rigid body evolution described in section 2.4, two basic assumptions are mandatory.

1) The flowing liquid is essentially undisturbed by the particle.

This assumption is reasonable if, for $\varepsilon \to 0$, the mass of the small body and thus its momentum transfer to the fluid is negligible (which is the case for $\varrho = \rho_b/\rho_f = \mathcal{O}(1)$).

2) The fluid motion induces a rotation of the rigid body with angular velocity $\boldsymbol{\omega} = \mathcal{O}(1)$.

From figure 2 we see that the velocity roughly varies between $\boldsymbol{u}_0 - \varepsilon\boldsymbol{\omega}\wedge\boldsymbol{p}$ and $\boldsymbol{u}_0 + \varepsilon\boldsymbol{\omega}\wedge\boldsymbol{p}$ along a distance of order ε. Consequently, the local velocity field around the particle (i.e. the difference to the undisturbed flow $\boldsymbol{u}_0$) has a gradient of order one while its magnitude is of order ε. In contrast to this, the local pressure has to be of order one. Otherwise it cannot balance the viscous forces which are proportional to the symmetric part of the gradient of $\boldsymbol{u}$.

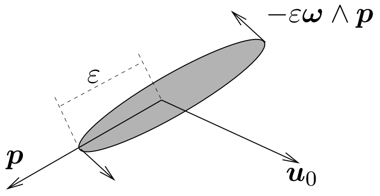

Figure 2: Two dimensional projection of the rigid body $E_\varepsilon(t)$ moving essentially with the undisturbed velocity $\boldsymbol{u}_0$ plus an $\mathcal{O}(\varepsilon)$ disturbance due to an angular velocity $\boldsymbol{\omega}$ of order one. The gradient of the local velocity field is of order one.

To describe the local velocity and pressure field, it is convenient to work in the fixed body coordinates $\boldsymbol{y}$. If $\varepsilon \boldsymbol{u}_1(t, \boldsymbol{y})$ is such a local velocity field (thought of as a perturbation from free flow $\boldsymbol{u}_0$), the actual velocity (perturbation) in $\boldsymbol{x}$ coordinates is $\varepsilon R(t)\boldsymbol{u}_1(t, \boldsymbol{Y}(t, \boldsymbol{x}, \varepsilon))$, where $\boldsymbol{Y}$ is given by (17). Assuming that $\boldsymbol{u}_1 = \mathcal{O}(1)$, we have exactly the situation that $\varepsilon \boldsymbol{u}_1$ is of order ε but the $\boldsymbol{x}$-gradient is of order one since $\nabla_{\boldsymbol{x}} \boldsymbol{Y} = \mathcal{O}(\varepsilon^{-1})$. Similarly, the leading order local pressure field is assumed of the form $p_1(t, \boldsymbol{Y}(t, \boldsymbol{x}, \varepsilon))$. Altogether, we use the following ansatz for the local fields

$$\begin{aligned} \boldsymbol{u}_{loc}(t, \boldsymbol{x}) &= \varepsilon R(t)\boldsymbol{u}_1(t, \boldsymbol{Y}(t, \boldsymbol{x}, \varepsilon)) + \varepsilon^2 R(t)\boldsymbol{u}_2(t, \boldsymbol{Y}(t, \boldsymbol{x}, \varepsilon)) + \dots, \\ p_{loc}(t, \boldsymbol{x}) &= p_1(t, \boldsymbol{Y}(t, \boldsymbol{x}, \varepsilon)) + \varepsilon p_2(t, \boldsymbol{Y}(t, \boldsymbol{x}, \varepsilon)) + \dots. \end{aligned} \tag{24}$$

If we assume polynomial decay rates for the local fields it turns out that the particle has a small global influence. For example, if $|\boldsymbol{u}_1(t, \boldsymbol{y})| \approx C_1|\boldsymbol{y}|^{-1} + C_2|\boldsymbol{y}|^{-2} + \dots$ for large $|\boldsymbol{y}|$, we have

$$|\varepsilon R(t)\boldsymbol{u}_1(t, \boldsymbol{Y}(t, \boldsymbol{x}, \varepsilon))| \approx \varepsilon^2 C_1|\boldsymbol{x} - \boldsymbol{c}(t)|^{-1} + \varepsilon^3 C_2|\boldsymbol{x} - \boldsymbol{c}(t)|^{-2} + \dots$$

Assuming that the particle stays away from the boundary, we see that the far field of the local velocity influences the velocity distribution at $\partial\Omega$ in different ε orders starting at order ε^2. Since we are aiming at most at second order accuracy, we can thus neglect higher order global fields (which ought to be included in a more accurate treatment). Thus, we end up with expansions of the form

$$\begin{aligned} \boldsymbol{u}(t, \boldsymbol{x}) &= \boldsymbol{u}_0(t, \boldsymbol{x}) + \varepsilon R(t)\boldsymbol{u}_1(t, \boldsymbol{Y}(t, \boldsymbol{x}, \varepsilon)) + \varepsilon^2 R(t)\boldsymbol{u}_2(t, \boldsymbol{Y}(t, \boldsymbol{x}, \varepsilon)) + \dots, \\ p(t, \boldsymbol{x}) &= p_0(t, \boldsymbol{x}) + p_1(t, \boldsymbol{Y}(t, \boldsymbol{x}, \varepsilon)) + \varepsilon p_2(t, \boldsymbol{Y}(t, \boldsymbol{x}, \varepsilon)) + \dots, \\ \boldsymbol{c}(t) &= \boldsymbol{c}_0(t) + \varepsilon \boldsymbol{c}_1(t) + \varepsilon^2 \boldsymbol{c}_2(t) + \dots, \\ \boldsymbol{\omega}(t) &= \boldsymbol{\omega}_0(t) + \varepsilon \boldsymbol{\omega}_1(t) + \dots, \\ R(t) &= R_0(t) + \varepsilon R_1(t) + \dots. \end{aligned}$$

To obtain reasonable equations for the coefficients $\boldsymbol{u}_i, p_i, \boldsymbol{c}_i, \boldsymbol{\omega}_i$ and R_i, the ansatz is inserted into the equations listed in section 2.4, Taylor expansions are carried out for $\varepsilon \to 0$, and the appearing expressions in different orders of ε are equated to zero separately. For reasons of clarity, we will skip this step but use its result to define the expansion coefficients. In appendix B, we show that the corresponding truncated expansion satisfies the original problem at least up to order $\mathcal{O}(\varepsilon)$ which supports its validity.

Finally, we want to stress that our expansion of the flow variables is only reasonable as long as the rigid body stays away from the boundary $\partial\Omega$ because we assume the functions $\boldsymbol{u}_i$, p_i to be defined in the unbounded exterior of the body template E. Once the distance to the boundary is of the order of ε, this assumption does not include the relevant physical effects.

3.2 Expansion coefficients

Here, we present the result of the asymptotic analysis outlined in the previous section. We define the coefficients of the flow fields and the rigid body variables according to the equations following from the expansion. The validity of the equations is checked, a posteriori, in appendix B.
The leading order coefficients $\boldsymbol{u}_0$, p_0 are defined as solutions of the incompressible Navier-Stokes problem in Ω

$$\mathrm{Re}(\partial_t \boldsymbol{u}_0 + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0) + \nabla p_0 = \Delta \boldsymbol{u}_0, \qquad \mathrm{div}\boldsymbol{u}_0 = 0 \tag{25}$$

with the same initial and boundary values as the full problem in Section 2.4. The body center of mass is, at leading order, determined by

$$\dot{\boldsymbol{c}}_0(t) = \boldsymbol{u}_0(t, \boldsymbol{c}_0(t)), \qquad \boldsymbol{c}_0(0) = \boldsymbol{c}(0). \tag{26}$$

The higher order perturbations $\boldsymbol{u}_i$, p_i, $i = 1, 2$ are determined as solutions of stationary Stokes problems in the exterior of the body template E

$$\nabla p_i = \Delta \boldsymbol{u}_i, \quad \mathrm{div}\boldsymbol{u}_i = 0, \qquad \text{in } E^c \tag{27}$$

with sufficiently fast decay at infinity ($\mathcal{O}(|\boldsymbol{y}|^{-1})$ for velocity and $\mathcal{O}(|\boldsymbol{y}|^{-2})$ for pressure and velocity gradient). At the body surface, we find integral conditions on the fluid stresses $\sigma_i = -p_i I + 2S[\boldsymbol{u}_i]$

$$\int_{\partial E} \sigma_i \boldsymbol{n}\, dS = \boldsymbol{g}_i, \qquad \int_{\partial E} \boldsymbol{y} \wedge \sigma_i \boldsymbol{n}\, dS = \boldsymbol{G}_i \tag{28}$$

and a Dirichlet condition

$$\boldsymbol{u}_i = \boldsymbol{b}_i = R_0^T \dot{\boldsymbol{X}}_i + \boldsymbol{H}_i \qquad \text{on } \partial E. \tag{29}$$

The functions $\boldsymbol{b}_i, \boldsymbol{g}_i, \boldsymbol{G}_i$ and $\boldsymbol{H}_i = \boldsymbol{b}_i - R_0^T \dot{\boldsymbol{X}}_i$ have been introduced to avoid confusing details which cloud the basic structure of the problem. They generally depend on lower order expansion coefficients and are given in detail below. The $\boldsymbol{X}_i$ are the coefficients in the expansion of $\boldsymbol{X}(t, \boldsymbol{y}, \varepsilon) = \varepsilon R(t)\boldsymbol{y} + \boldsymbol{c}(t)$, i.e.

$$\boldsymbol{X}_0 = \boldsymbol{c}_0, \quad \boldsymbol{X}_1 = R_0 \boldsymbol{y} + \boldsymbol{c}_1, \quad \boldsymbol{X}_2 = R_1 \boldsymbol{y} + \boldsymbol{c}_2, \ldots \tag{30}$$

Finally, the matrices R_0, R_1 satisfy the differential equations

$$\begin{aligned} \dot{R}_0 &= B(\boldsymbol{\omega}_0) R_0, & R_0(0) &= R(0), \\ \dot{R}_1 &= B(\boldsymbol{\omega}_0) R_1 + B(\boldsymbol{\omega}_1) R_0, & R_1(0) &= 0 \end{aligned} \tag{31}$$

and the initial values for $\boldsymbol{c}_i$ are

$$\boldsymbol{c}_1(0) = 0, \qquad \boldsymbol{c}_2(0) = 0. \tag{32}$$

To write the functions $\boldsymbol{b}_i, \boldsymbol{g}_i, \boldsymbol{G}_i$ in a compact form we use the differential operators D_i which appear in the Taylor expansion of an expression $f(\boldsymbol{X}_0 + \varepsilon \boldsymbol{X}_1 + \varepsilon^2 \boldsymbol{X}_2 + \dots)$ with respect to ε. More precisely, D_i are defined by formally equating orders in

$$\sum_{i=0}^{\infty} \varepsilon^i (D_i f)(\boldsymbol{X}_0) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\sum_{i=1}^{\infty} \varepsilon^i (\boldsymbol{X}_i \cdot \nabla) \right]^k f(\boldsymbol{X}_0)$$

so that D_i depends on $\boldsymbol{X}_0, \dots, \boldsymbol{X}_i$. For our purpose, we need

$$D_0 = 1, \quad D_1 = \boldsymbol{X}_1 \cdot \nabla, \quad D_2 = \boldsymbol{X}_2 \cdot \nabla + \frac{1}{2} (\boldsymbol{X}_1 \cdot \nabla)^2.$$

In the first Stokes problem, we have

$$\boldsymbol{g}_1 = 0, \quad \boldsymbol{G}_1 = 0, \quad \boldsymbol{b}_1 = R_0^T (\dot{\boldsymbol{X}}_1 - D_1 \boldsymbol{u}_0). \tag{33}$$

Here and in the following, the derivatives of $\boldsymbol{u}_0$ are evaluated at time t and position $\boldsymbol{c}_0(t)$. The second Stokes problem is specified by

$$\boldsymbol{b}_2 = R_0^T (\dot{\boldsymbol{X}}_2 - D_2 \boldsymbol{u}_0 - R_1 \boldsymbol{u}_1), \tag{34}$$

$$\boldsymbol{g}_2 = \varrho |E| \mathrm{Re} R_0^T \ddot{\boldsymbol{c}}_0 - \int_{\partial E} R_0^T D_1 \sigma_0 R_0 \boldsymbol{n} \, dS, \tag{35}$$

$$\boldsymbol{G}_2 = - \int_{\partial E} \boldsymbol{y} \wedge (R_0^T D_1 \sigma_0 R_0 \boldsymbol{n}) \, dS. \tag{36}$$

3.3 Solvability conditions

At first glance, the Stokes problems for $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ seem to be overdetermined because of the extra requirements (28) apart from the Dirichlet conditions (29). However, a closer inspection reveals that the Dirichlet conditions (29) are not fully determined because they depend on the time derivative of the rigid body variables $\boldsymbol{c}_i$ and R_{i-1} through $\dot{\boldsymbol{X}}_i$. Moreover, the number of degrees of freedom hidden in $\dot{\boldsymbol{c}}_i$ and $\dot{R}_{i-1}$ match exactly the number of integral conditions. Note that the vector $\dot{\boldsymbol{c}}_i$ has three components and the time derivative of R_{i-1} is, in view of (31), also determined by three free parameters. In fact, we can summarize (31) as

$$\dot{R}_i = B(\boldsymbol{\omega}_i) R_0 + K_i \tag{37}$$

where the matrix K_i depends on lower order terms and on R_i itself so that the equation for R_i is completely known once the three components of $\boldsymbol{\omega}_i$ are determined.

We now show that there is only one choice for the six quantities such that both (28) and (29) are satisfied. These relations for $\dot{\boldsymbol{c}}_i$, $\dot{R}_{i-1}$ can therefore be viewed as solvability conditions for the Stokes problem. The basis for the proof is the so called Faxén's law [7,10,13] which relates the Dirichlet values $\boldsymbol{b}_i$ of velocity with the averaged force integrals $\boldsymbol{g}_i$ and $\boldsymbol{G}_i$. Following [13], the idea is as follows: we first construct particular solutions $\boldsymbol{w}_1, \dots, \boldsymbol{w}_6$ of the Stokes equation in E^c without source term and with Dirichlet boundary conditions

$$\boldsymbol{w}_k = \boldsymbol{e}_k \quad k = 1, 2, 3, \qquad \boldsymbol{w}_k = \boldsymbol{y} \wedge \boldsymbol{e}_k \quad k = 4, 5, 6, \qquad \text{on } \partial E$$

where $\boldsymbol{e}_4 = \boldsymbol{e}_1$, $\boldsymbol{e}_5 = \boldsymbol{e}_2$ and $\boldsymbol{e}_6 = \boldsymbol{e}_3$. The stress tensors corresponding to $\boldsymbol{w}_k$ are denoted $\sigma[\boldsymbol{w}_k]$. Then (28) implies

$$\boldsymbol{e}_k \cdot \boldsymbol{g}_i = \int_{\partial E} \boldsymbol{w}_k \cdot \sigma_i \boldsymbol{n} \, dS, \qquad k = 1, 2, 3.$$

Using the Green's formula for the Stokes equation (see appendix C), it follows with the boundary condition (29)

$$\boldsymbol{e}_k \cdot \boldsymbol{g}_i = \int_{\partial E} \boldsymbol{b}_i \cdot \sigma[\boldsymbol{w}_k] \boldsymbol{n} \, dS, \qquad k = 1, 2, 3. \tag{38}$$

Similarly, we have

$$\boldsymbol{e}_k \cdot \boldsymbol{G}_i = \int_{\partial E} (B(\boldsymbol{y}) \sigma_i \boldsymbol{n}) \cdot \boldsymbol{e}_k \, dS = -\int_{\partial E} (\sigma_i \boldsymbol{n}) \cdot (B(\boldsymbol{y}) \boldsymbol{e}_k) \, dS, \qquad k = 4, 5, 6$$

where the skew symmetry of B has been used in the last equality. Since $\boldsymbol{w}_k = B(\boldsymbol{y})\boldsymbol{e}_k$ on ∂E, we obtain again with the help of Green's formula

$$\boldsymbol{e}_k \cdot \boldsymbol{G}_i = -\int_{\partial E} \boldsymbol{b}_i \cdot \sigma[\boldsymbol{w}_k] \boldsymbol{n} \, dS \qquad k = 4, 5, 6. \tag{39}$$

We now replace $\boldsymbol{b}_i$ by the more detailed structure given in equation (29). First, we note that (30) implies in connection with (37)

$$\dot{\boldsymbol{X}}_i = \dot{R}_{i-1} \boldsymbol{y} + \dot{\boldsymbol{c}}_i = B(\boldsymbol{\omega}_{i-1}) R_0 \boldsymbol{y} + K_{i-1} \boldsymbol{y} + \dot{\boldsymbol{c}}_i.$$

Observing that for any rotation matrix R and any vector $\boldsymbol{\omega}$ we have relation (55), i.e. $R^T B(\boldsymbol{\omega}) R = B(R^T \boldsymbol{\omega})$, it follows

$$R_0^T \dot{\boldsymbol{X}}_i = B(R_0^T \boldsymbol{\omega}_{i-1}) \boldsymbol{y} + R_0^T \dot{\boldsymbol{c}}_i + R_0^T K_{i-1} \boldsymbol{y} = -B(\boldsymbol{y}) R_0^T \boldsymbol{\omega}_{i-1} + R_0^T \dot{\boldsymbol{c}}_i + R_0^T K_{i-1} \boldsymbol{y},$$

so that, using skew symmetry of B,

$$\begin{aligned} \boldsymbol{b}_i \cdot (\sigma[\boldsymbol{w}_k] \boldsymbol{n}) = (R_0^T \boldsymbol{\omega}_{i-1}) \cdot (\boldsymbol{y} \wedge \sigma[\boldsymbol{w}_k] \boldsymbol{n}) + (R_0^T \dot{\boldsymbol{c}}_i) \cdot (\sigma[\boldsymbol{w}_k] \boldsymbol{n}) \\ + (R_0^T K_{i-1} \boldsymbol{y} + \boldsymbol{H}_i) \cdot (\sigma[\boldsymbol{w}_k] \boldsymbol{n}) \end{aligned} \tag{40}$$

Inserting (40) into (38) and (39) we obtain with the abbreviation

$$\boldsymbol{F}_k = \int_{\partial E} \sigma[\boldsymbol{w}_k] \boldsymbol{n} \, dS, \qquad \boldsymbol{M}_k = \int_{\partial E} \boldsymbol{y} \wedge \sigma[\boldsymbol{w}_k] \boldsymbol{n} \, dS \tag{41}$$

the following equations

$$\boldsymbol{F}_k \cdot (R_0^T \dot{\boldsymbol{c}}_i) + \boldsymbol{M}_k \cdot (R_0^T \boldsymbol{\omega}_{i-1}) = A_{ki}, \quad k = 1, \dots, 6 \tag{42}$$

where all terms independent of $\boldsymbol{\omega}_{i-1}$ and $\dot{\boldsymbol{c}}_i$ are collected in A_{ki}

$$A_{ki} = -\int_{\partial E} (R_0^T K_{i-1}\boldsymbol{y} + \boldsymbol{H}_i) \cdot \sigma[\boldsymbol{w}_k]\boldsymbol{n}\, dS + \begin{cases} \boldsymbol{e}_k \cdot \boldsymbol{g}_i & k = 1,2,3 \\ -\boldsymbol{e}_k \cdot \boldsymbol{G}_i & k = 4,5,6 \end{cases}. \qquad (43)$$

From the system (42) we can derive six independent explicit equations for the six unknowns $\dot{\boldsymbol{c}}_i$ and $\boldsymbol{\omega}_{i-1}$ (i.e. ordinary differential equations for $\boldsymbol{c}_i$ and R_{i-1}) because the vectors $\boldsymbol{L}_k = \left(\begin{smallmatrix} \boldsymbol{F}_k \\ -\boldsymbol{M}_k \end{smallmatrix}\right)$ are linearly independent. According to the definition of $\boldsymbol{F}_k$, $\boldsymbol{M}_k$, the components of $\boldsymbol{L}_k$ are given by

$$(\boldsymbol{L}_k)_j = \int_{\partial E} \boldsymbol{w}_j \cdot (\sigma[\boldsymbol{w}_k]\boldsymbol{n})\, dS.$$

The proof of linear independence is given in appendix C, Lemma 8.
Altogether, we end up with the following pattern to determine the expansion coefficients: first, $\boldsymbol{u}_0, p_0, \boldsymbol{c}_0$ are calculated from (25) and (26). Then, $\dot{\boldsymbol{c}}_1, \boldsymbol{\omega}_0$ are obtained using the solvability condition (42) and $\boldsymbol{c}_1, R_0$ follow by integrating the resulting ordinary differential equations. In the next step, the Stokes problem (27) with Dirichlet conditions (29) is solved to obtain $\boldsymbol{u}_1, p_1$. The same pattern applies to the evaluation of $\boldsymbol{u}_2, p_2, \boldsymbol{c}_2, R_1$.

4 Extracting Jeffery's equation

The aim of this section is to show that the solvability conditions (42) for $\dot{\boldsymbol{c}}_1$ and $\boldsymbol{\omega}_0$ give rise to Jeffery's equation in the special case of ellipsoidal bodies.
Let us therefore consider (42) for the case $i = 1$. Since $\boldsymbol{g}_1 = \boldsymbol{G}_1 = 0$, $K_0 = 0$ (see (37)) and $\boldsymbol{H}_1 = \boldsymbol{b}_1 - R_0^T \dot{\boldsymbol{X}}_1 = -R_0^T D_1 \boldsymbol{u}_0$ (see (29) and (33)), we have

$$A_{k1} = \int_{\partial E} (R_0^T D_1 \boldsymbol{u}_0) \cdot \sigma[\boldsymbol{w}_k]\boldsymbol{n}\, dS.$$

By observing that $D_1 = \boldsymbol{X}_1 \cdot \nabla = (R_0\boldsymbol{y} + \boldsymbol{c}_1) \cdot \nabla$, we see that

$$R_0^T D_1 \boldsymbol{u}_0 = R_0^T \nabla \boldsymbol{u}_0 (R_0 \boldsymbol{y} + \boldsymbol{c}_1)$$

where all $\boldsymbol{u}_0$ derivatives are evaluated at $(t, \boldsymbol{c}_0(t))$. Splitting the gradient of $\boldsymbol{u}_0$ into symmetric and skew symmetric part and observing (55) and (58), we find

$$R_0^T D_1 \boldsymbol{u}_0 = B(R_0^T \operatorname{curl} \boldsymbol{u}_0/2)\boldsymbol{y} + (R_0^T S[\boldsymbol{u}_0] R_0)\boldsymbol{y} + R_0^T \nabla \boldsymbol{u}_0 \boldsymbol{c}_1$$

and since $B(\boldsymbol{\alpha})\boldsymbol{\beta} = B^T(\boldsymbol{\beta})\boldsymbol{\alpha}$,

$$A_{k1} = \left(\frac{1}{2} R_0^T \operatorname{curl} \boldsymbol{u}_0\right) \cdot \int_{\partial E} B(\boldsymbol{y}) \sigma[\boldsymbol{w}_k]\boldsymbol{n}\, dS + (R_0^T \nabla \boldsymbol{u}_0 \boldsymbol{c}_1) \cdot \int_{\partial E} \sigma[\boldsymbol{w}_k]\boldsymbol{n}\, dS$$
$$+ \int_{\partial E} (R_0^T S[\boldsymbol{u}_0] R_0)\boldsymbol{y} \cdot \sigma[\boldsymbol{w}_k]\boldsymbol{n}\, dS$$

With the definition (41) we can cast the solvability conditions (42) for $i = 1$ in the form

$$\begin{aligned}\boldsymbol{F}_k \cdot R_0^T(\dot{\boldsymbol{c}}_1 - \nabla \boldsymbol{u}_0 \boldsymbol{c}_1) + \boldsymbol{M}_k \cdot R_0^T(\boldsymbol{\omega}_0 - \operatorname{curl} \boldsymbol{u}_0/2) \\ = (R_0^T S[\boldsymbol{u}_0] R_0) : \int_{\partial E} \boldsymbol{y} \otimes \sigma[\boldsymbol{w}_k]\boldsymbol{n}\, dS, \quad k = 1, \ldots, 6 \end{aligned} \tag{44}$$

where $(\boldsymbol{\alpha} \otimes \boldsymbol{\beta})_{ij} = \alpha_i \beta_j$ and $A : B = A_{ij} B_{ij}$. We observe that only zero and first order moments of the surface force $\sigma[\boldsymbol{w}_k]\boldsymbol{n}$ are required to evaluate the coefficients of (44). Note that in this way, geometry information about the rigid body which is coded in the Stokes fields $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_6$, enters the equation for $\boldsymbol{c}_1$ and $\boldsymbol{\omega}_0$. For general shapes of E, the moments can be calculated, for example, with a boundary element method. However, in the special case of *ellipsoidal bodies*, an explicit representation of $\sigma[\boldsymbol{w}_k]\boldsymbol{n}$ is available. In the following, we concentrate on this case.

4.1 The case of ellipsoidal bodies

Let $B_1 = \{\boldsymbol{z} \in \mathbb{R}^3 : |\boldsymbol{z}| < 1\}$ denote the unit ball in $\mathbb{R}^3$. A general axis parallel ellipsoid is then given by

$$E = DB_1, \qquad D = \operatorname{diag}(d_1, d_2, d_3), \quad d_i > 0.$$

Note that $\boldsymbol{y} \in E$ if and only if $z = D^{-1}\boldsymbol{y} \in B_1$, i.e. if

$$\left(\frac{y_1}{d_1}\right)^2 + \left(\frac{y_2}{d_2}\right)^2 + \left(\frac{y_3}{d_3}\right)^2 < 1.$$

Clearly, the center of mass is at the origin and the volume is given by $|E| = |B_1| \det D$.
Combining the result in [13] with the expression (78) in appendix D for the surface element, we can express the surface force on the ellipsoid as

$$(\sigma[\boldsymbol{w}_k]\boldsymbol{n})(\boldsymbol{y})\, dS(\boldsymbol{y}) = \alpha_k \det D\, \boldsymbol{w}_k(D\boldsymbol{z})\, dS(\boldsymbol{z}), \qquad \boldsymbol{y} = D\boldsymbol{z} \in \partial E \tag{45}$$

where α_k are non-zero constants. Using this relation, it is possible to evaluate all required surface moments and the unspecified constants α_k drop out in the end because they appear on both sides of (44). The detailed computation of the moments is given in appendix D. Here, we only list the results.
The surface moments on the right hand side of (44) are given by

$$\int_{\partial E} \boldsymbol{y} \otimes \sigma[\boldsymbol{w}_k]\boldsymbol{n}\, dS = \begin{cases} 0 & k = 1, 2, 3, \\ \alpha_k |E| D^2 B(\boldsymbol{e}_k) & k = 4, 5, 6. \end{cases}$$

and for $\boldsymbol{F}_k$ and $\boldsymbol{M}_k$, we find

$$\boldsymbol{F}_k = \begin{cases} 3\alpha_k |E| \boldsymbol{e}_k & k = 1, 2, 3, \\ 0 & k = 4, 5, 6 \end{cases} \qquad \boldsymbol{M}_k = \begin{cases} 0 & k = 1, 2, 3, \\ -\alpha_k |E| T\boldsymbol{e}_k & k = 4, 5, 6 \end{cases}$$

where T is the inertia tensor of E defined in (23)

$$T = \begin{pmatrix} d_2^2 + d_3^2 & & \\ & d_1^2 + d_3^2 & \\ & & d_1^2 + d_2^2 \end{pmatrix}.$$

Inserting these results into (44), the solvability conditions decouple

$$\begin{aligned} &\dot{\boldsymbol{c}}_1 = \nabla \boldsymbol{u}_0 \boldsymbol{c}_1, \\ -&\boldsymbol{e}_k \cdot T R_0^T (\boldsymbol{\omega}_0 - \operatorname{curl} \boldsymbol{u}_0/2) = R_0^T S[\boldsymbol{u}_0] R_0 : D^2 B(\boldsymbol{e}_k), \quad k = 1,2,3 \end{aligned}$$

where the $\boldsymbol{u}_0$ expressions are evaluated at $(t, \boldsymbol{c}_0(t))$. Since the $\boldsymbol{c}_1$ equation is homogeneous, the zero initial condition (32) implies $\boldsymbol{c}_1(t) = 0$ for all t, i.e. the solvability conditions take the form $\boldsymbol{c}_1 = 0$ and

$$-\boldsymbol{e}_k \cdot T R_0^T (\boldsymbol{\omega}_0 - \operatorname{curl} \boldsymbol{u}_0/2) = R_0^T S[\boldsymbol{u}_0] R_0 : D^2 B(\boldsymbol{e}_k), \quad k = 1,2,3. \tag{46}$$

To see the relation between the equations for $\boldsymbol{\omega}_0$ and Jeffery's equation, some further transformations are necessary. We recall that $\boldsymbol{\omega}_0$ is required to set up the differential equation

$$\dot{R}_0 = B(\boldsymbol{\omega}_0) R_0, \quad R_0(0) = R(0)$$

for the leading order rigid body rotation R_0. This matrix differential equation can equivalently be written as three vector differential equations for the columns $\boldsymbol{p}_{0i} = R_0 \boldsymbol{e}_i$. Note that each of these vectors points in the directions of a principal axis. In appendix D we show that the three orientation vectors satisfy the differential equations

$$\dot{\boldsymbol{p}}_{0i} = \frac{1}{2} (\operatorname{curl} \boldsymbol{u}_0) \wedge \boldsymbol{p}_{0i} + \sum_{k,m} \epsilon_{ikm} \lambda_m \boldsymbol{p}_{0k} \otimes \boldsymbol{p}_{0k} S[\boldsymbol{u}_0] \boldsymbol{p}_{0i}, \qquad i = 1,2,3 \tag{47}$$

where the tensor ϵ_{ikm} is defined in section A.1 and the parameters λ_m are given by the ratios

$$\lambda_1 = \frac{d_2^2 - d_3^2}{d_3^2 + d_2^2}, \qquad \lambda_2 = \frac{d_3^2 - d_1^2}{d_1^2 + d_3^2}, \qquad \lambda_3 = \frac{d_1^2 - d_2^2}{d_2^2 + d_1^2}.$$

Equation (47) is supplemented by initial conditions $\boldsymbol{p}_{0i}(0) = R(0)\boldsymbol{e}_i$ according to (31).

In the particular case where the ellipsoid is a body of rotation (i.e. $d_1 = l$ and $d_2 = d_3 = d$), the orientation vectors $\boldsymbol{p}_{02}, \boldsymbol{p}_{03}$ are not required to specify the spatial orientation of the body. They only describe how much the body has rotated around the axis $\boldsymbol{p}_{01}$. This is nicely reflected by the fact that the equation for $\boldsymbol{p}_{01}$ decouples from the other two equations in that case. To see this, we note that

$$\lambda_1 = 0, \qquad \lambda_3 = -\lambda_2 = \lambda = \frac{(l/d)^2 - 1}{(l/d)^2 + 1},$$

and the equation for the orientation vector $\boldsymbol{p}_{01}$ has the form

$$\dot{\boldsymbol{p}}_{01} = \frac{1}{2}(\operatorname{curl} \boldsymbol{u}_0) \wedge \boldsymbol{p}_{01} + \lambda \sum_{k=2}^{3} \boldsymbol{p}_{0k} \otimes \boldsymbol{p}_{0k} S[\boldsymbol{u}_0]\boldsymbol{p}_{01}.$$

Taking into account that

$$\sum_{k=1}^{3} \boldsymbol{p}_{0k} \otimes \boldsymbol{p}_{0k} = R_0 \left(\sum_{k=1}^{3} \boldsymbol{e}_k \otimes \boldsymbol{e}_k \right) R_0^T = R_0 I R_0^T = I,$$

we can write

$$\sum_{k=2}^{3} \boldsymbol{p}_{0k} \otimes \boldsymbol{p}_{0k} S[\boldsymbol{u}_0]\boldsymbol{p}_{01} = S[\boldsymbol{u}_0]\boldsymbol{p}_{01} - \boldsymbol{p}_{01} \otimes \boldsymbol{p}_{01} S[\boldsymbol{u}_0]\boldsymbol{p}_{01}$$

and hence

$$\dot{\boldsymbol{p}}_{01} = \frac{1}{2}(\operatorname{curl} \boldsymbol{u}_0) \wedge \boldsymbol{p}_{01} + \lambda(S[\boldsymbol{u}_0]\boldsymbol{p}_{01} - \boldsymbol{p}_{01} \otimes \boldsymbol{p}_{01} S[\boldsymbol{u}_0]\boldsymbol{p}_{01}).$$

Since the cubic term $\boldsymbol{p}_{01} \otimes \boldsymbol{p}_{01} S[\boldsymbol{u}_0]\boldsymbol{p}_{01}$ can be rewritten as $(\boldsymbol{p}_{01}^T S[\boldsymbol{u}_0]\boldsymbol{p}_{01})\boldsymbol{p}_{01}$, we obtain upon renaming $\boldsymbol{p}_0 = \boldsymbol{p}_{01}$

$$\dot{\boldsymbol{p}}_0 = \frac{1}{2}(\operatorname{curl} \boldsymbol{u}_0) \wedge \boldsymbol{p}_0 + \lambda(S[\boldsymbol{u}_0]\boldsymbol{p}_0 - (\boldsymbol{p}_0^T S[\boldsymbol{u}_0]\boldsymbol{p}_0)\boldsymbol{p}_0).$$

This is exactly Jeffery's equation (2) which thus turns out to be the leading order solvability condition in the case of an elongated, rotationally symmetric ellipsoidal body.

5 Some solutions

In this section, we try to illuminate the behavior of Jeffery's equation in the case of several stationary linear flow fields $\boldsymbol{u}_0(\boldsymbol{x}) = A\boldsymbol{x}$. For several classes of matrices, explicit solutions of Jeffery's equation are known. We will not list these formulas here (they can be found, for example, in [8, 9]), but try to illustrate typical solutions.
Coming back to the linear flow field, we stress that $\boldsymbol{u}_0$ is a solution of the Navier-Stokes equation if and only if A^2 is symmetric and $\operatorname{tr}(A) = 0$ (the divergence condition). In this case the pressure is given by $p(\boldsymbol{x}) = -\boldsymbol{x}^T A^2 \boldsymbol{x}$. In two space dimensions the symmetry of A^2 is a consequence of the trace condition because the off-diagonal entries of A^2 are $A_{21}\operatorname{tr}(A)$ and $A_{12}\operatorname{tr}(A)$. For truly three-dimensional flows, however, the conditions are independent, as the following example shows

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Even though the linear flow field is not a solution of the Navier-Stokes equation if A^2 is not symmetric, it still satisfies the Stokes equation (together with a constant pressure). Looking back at the asymptotic expansion, it is clear that a similar derivation is possible if we start with Stokes instead of Navier-Stokes equation. In this case, the leading order flow field $\boldsymbol{u}_0$ is an undisturbed Stokes solution. In that sense, it may also be reasonable to consider flows with $A^2 \neq (A^2)^T$.
To explain our geometrical representation of the Jeffery solutions, let us begin with the simple case of a purely rotational flow

$$\boldsymbol{u}_0(\boldsymbol{x}) = A\boldsymbol{x}, \qquad A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, the symmetric part $S[\boldsymbol{u}_0]$ of the Jacobian vanishes and $\operatorname{curl} \boldsymbol{u}_0 = 2\boldsymbol{e}_3$, so that Jeffery's equation reduces to

$$\dot{\boldsymbol{p}}_0 = \boldsymbol{e}_3 \wedge \boldsymbol{p}_0, \qquad \boldsymbol{p}_0(0) = \bar{\boldsymbol{p}} \in S^2.$$

Consequently, the orientation vector (and thus the ellipsoid) performs a rotation around the $\boldsymbol{e}_3$-axis. In figure 3, the velocity field is shown in the $x_3 = 0$ plane together with a cut through an ellipsoid. The ratio between major axis of length l and minor axis of length d is given by the parameter $\lambda = ((l/d)^2 - 1)/((l/d)^2 + 1)$ according to

$$\frac{l}{d} = \sqrt{\frac{1+\lambda}{1-\lambda}}.$$

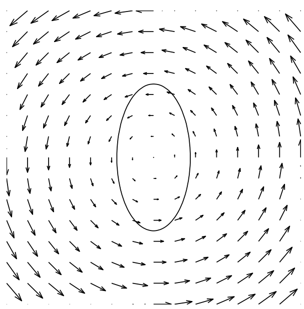

Figure 3: An ellipsoid with $\lambda = 0.6$ immersed in a rotational flow.

Since the rotation of the small body is determined by the local velocity field relative to the center of mass

$$\boldsymbol{u}_{rel}(t, \boldsymbol{y}) = \boldsymbol{u}_0(\boldsymbol{c}_0(t) + \boldsymbol{y}) - \boldsymbol{u}_0(\boldsymbol{c}_0(t)) = A\boldsymbol{y}$$

we can best imagine the behavior of the ellipsoid by drawing its center at the origin of the flow field. Note, however, that the center of mass $\boldsymbol{c}_0$ would follow a streamline of the field depending on the initial position $\boldsymbol{c}_0(0)$ while the rotation takes place as if the ellipsoid was attached to the origin (i.e. $\boldsymbol{c}_0(0) = 0$).

In order to show the dynamical behavior of the orientation vector, we can plot its path on the unit sphere. In figure 4, the trajectories for several initial conditions $\boldsymbol{p}_0(0) = \bar{\boldsymbol{p}}$ are shown. In each case, the initial orientation is indicated by a little pin. The markers along the curve are equidistant in time. As expected, the trajectories are lines of equal latitude on the sphere because the rotation takes place around the north-south axis.

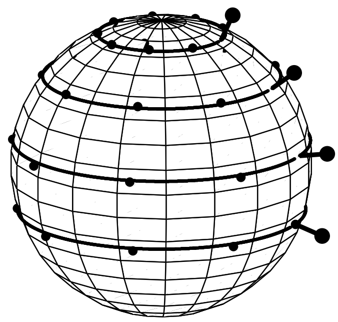

Figure 4: Orientation trajectories in a rotational flow field ($\lambda = 0.6$).

The next example concerns a flow field which stretches $\boldsymbol{e}_1$ direction and compresses in $\boldsymbol{e}_2$ direction

$$\boldsymbol{u}_0(\boldsymbol{x}) = A\boldsymbol{x}, \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{48}$$

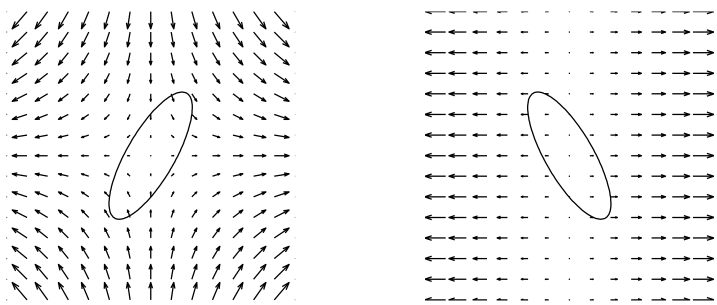

Figure 5: An ellipsoid with $\lambda = 0.8$ immersed in a stretching flow. Left: flow field in (x, y) plane. Right: flow field in (x, z) plane.

Since A is symmetric, there is no rotational part in Jeffery's equation

$$\dot{\boldsymbol{p}}_0 = \lambda(A\boldsymbol{p}_0 - (\boldsymbol{p}_0^T A\boldsymbol{p}_0)\boldsymbol{p}_0), \qquad \boldsymbol{p}_0(0) = \bar{\boldsymbol{p}} \in S^2. \tag{49}$$

In order to find stationary solutions we consider the case of general symmetric flow matrices A. The condition for stationary states

$$A\bar{\boldsymbol{p}} - (\bar{\boldsymbol{p}}^T A\bar{\boldsymbol{p}})\bar{\boldsymbol{p}} = 0, \qquad \bar{\boldsymbol{p}} \in S^2$$

implies that $\bar{\boldsymbol{p}}$ is a normalized eigenvector of A with eigenvalue $\bar{\boldsymbol{p}}^T A\bar{\boldsymbol{p}}$. In the following, we denote the eigenvectors by $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ and the corresponding eigenvalues as μ_i.
To check the stability of the stationary solutions, we calculate the derivative of the right hand side of (49) with respect to $\boldsymbol{p}$. Observing that $A = \sum_i \mu_i \boldsymbol{p}_i \otimes \boldsymbol{p}_i$, the derivative at $\boldsymbol{p} = \boldsymbol{p}_k$ turns out to be

$$\sum_{i=1}^{3} (\mu_i - \mu_k - 2\mu_k \delta_{ik}) \boldsymbol{p}_i \otimes \boldsymbol{p}_i,$$

having the same eigenvectors as A. Obviously, the compressing directions ($\mu_k < 0$) are unstable, because the derivative at $\boldsymbol{p}_k$ has a positive eigenvalue $-2\mu_k > 0$. Conversely, if μ_k strictly dominates the other eigenvalues, the corresponding eigenvector is a stable state since $\mu_i - \mu_k < 0$ for all i. Another positive eigenvalue μ_j belongs to an unstable state in that case because $\mu_k - \mu_j > 0$. The situation where two positive eigenvalues are of the same size is special because there is no single direction associated to the stretching but a whole circle (the two-dimensional eigenspace intersected with the sphere). We consider this example later.
In the case (48), we expect all generic trajectories to converge to the stretching direction $\boldsymbol{e}_1$ (see figure 6). Intuitively, this is also evident from the flow field (figure 5).

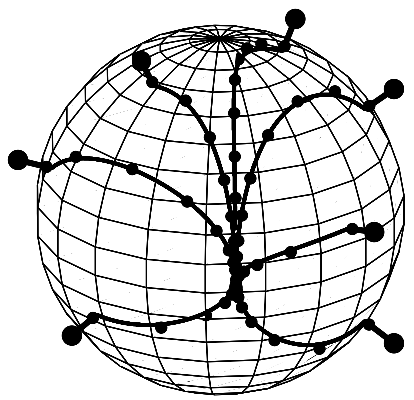

Figure 6: Trajectories of the orientation vector of an ellipsoid immersed in a stretching flow converge to the stretching direction ($\lambda = 0.8$).

Next let us turn to the case

$$\boldsymbol{u}_0(\boldsymbol{x}) = A\boldsymbol{x}, \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

where the positive eigenvalues are of the same size. The flow field in the (x, y) plane is presented in figure 7 which suggests that every orientation vector is simply pulled into the (x, y)-plane. This can also be seen from the trajectories.

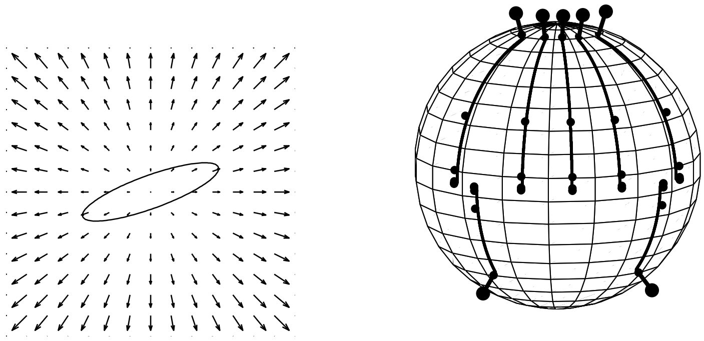

Figure 7: An ellipsoid with $\lambda = 0.9$ immersed in a stretching flow. Left: flow field in (x, y) plane. Right: trajectories of the orientation vector.

The shear flow is a typical flow where rotation and stretching behavior is mixed. As example, we consider

$$\boldsymbol{u}_0(\boldsymbol{x}) = A\boldsymbol{x}, \qquad A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

for which the flow field is presented in figure 8. The odd-even decomposition of the flow gradient A is

$$A = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where the odd part describes a positively oriented rotation around the $\boldsymbol{e}_3$ axis and the even part is a stretching flow with eigenvalues $-1, 1$ and corresponding eigendirections $(1, 1, 0)^T$ and $(-1, 1, 0)^T$.

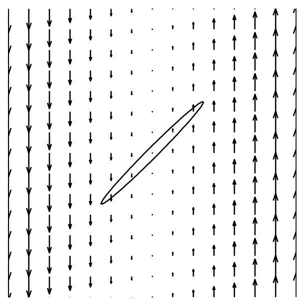

Figure 8: An ellipsoid with $\lambda = 0.99$ immersed in a stretching flow. Left: flow field in (x, y) plane. Right: flow field in (x, z) plane.

The corresponding Jeffery's solutions are periodic (see [9, 14] for analytical solutions). In view of the flow field it is clear that the ellipsoid moves fastest, if $\boldsymbol{p}$ is located in the (x, z) plane. But also in the orthogonal configuration, the shear flow leads to a rotation. Some trajectories are shown for different aspect ratios l/d in figure 9. Note that for large aspect ratios, the orientation vector spends most of its time close to the (y, z) plane which can be seen from the marker density along the trajectory. The reason is that the flow field along the sides of an ellipsoid pointing in y direction is very small. In the limit case $\lambda = 1$, where the ellipsoid collapses to a line segment ($d \to 0$), the y direction is a steady state.

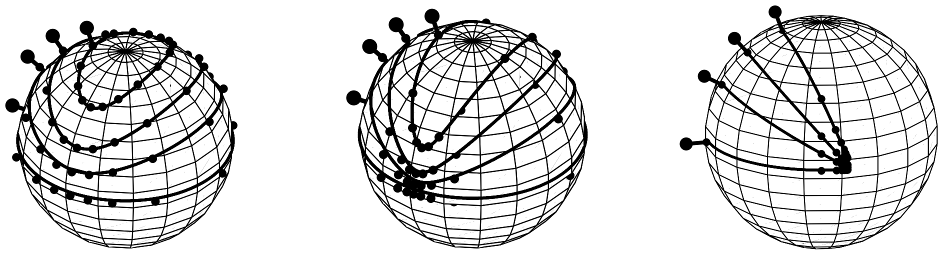

Figure 9: Orientation behavior in shear flow. Left: $\lambda = 0.7$. Middle: $\lambda = 0.95$. Right: $\lambda = 1.0$.

We close our considerations with a flow that combines shearing and stretching (and which is only a Stokes solution since $A^2 \neq (A^2)^T$).

$$\boldsymbol{u}_0(\boldsymbol{x}) = A\boldsymbol{x}, \qquad A = \begin{pmatrix} 0.1 & 0 & 1 \\ 0 & -0.1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this flow, the orientation vector oscillates around and converges to the stable equilibrium $(0, 1, 0)$.

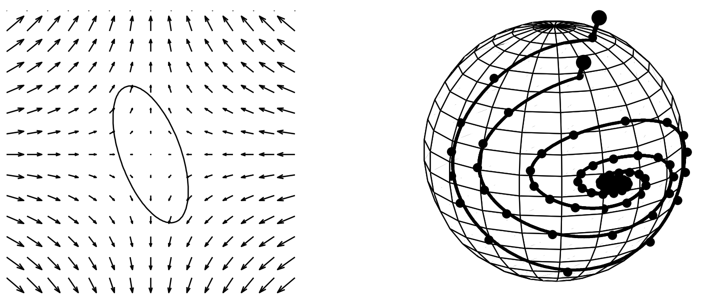

Figure 10: An ellipsoid with $\lambda = 0.7$ immersed in a shearing and stretching flow. Left: flow field in (x, y) plane. Right: orientation behavior.

A Rotations

Due to the non-linear structure of $SO(3)$, the asymptotic expansion of an ε-dependent rotation matrix

$$R = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \varepsilon^3 R_3 + \ldots$$

has the unpleasant property that any truncation at order $m \geq 1$ generally fails to be a rotation matrix. This leads to a few technicalities we want to address in section A.3. Apart from that, we prove some algebraic relations which are needed when manipulating skew symmetric matrices (section A.1). The relationship between rotations and skew symmetric matrices is highlighted in section A.2.

A.1 Skew symmetric matrices

The skew symmetric matrices in $\mathbb{R}^3$ can be parameterized in the basis $\epsilon_1, \epsilon_2, \epsilon_3$

$$\epsilon_1 = (\epsilon_{1jk}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \epsilon_2 = (\epsilon_{2jk}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{pmatrix},$$

and

$$\epsilon_3 = (\epsilon_{3jk}) = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $\epsilon_{ijk} = +1$ if i, j, k is an even permutation of $1, 2, 3$, $\epsilon_{ijk} = -1$ in the case of an odd permutation, and $\epsilon_{ijk} = 0$ in the case that some index appears twice. To abbreviate general linear combinations of the matrices ϵ_i, we introduce

$$B(\boldsymbol{\omega}) = \sum_{i=1}^{3} \omega_i \epsilon_i^T = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega} \in \mathbb{R}^3. \tag{50}$$

In the following, we employ Einstein's summation convention which allows us to write the previous relation simply as $B(\boldsymbol{\omega})_{jk} = \omega_i \epsilon_{ikj}$. The results below are direct consequences of definition (50).

Lemma 1 *Let $\boldsymbol{\omega}, \boldsymbol{x} \in \mathbb{R}^3$, $R \in SO(3)$, and $A \in \mathbb{R}^{3\times 3}$ be symmetric. Then*

$$B(\boldsymbol{\omega})\boldsymbol{x} = \boldsymbol{\omega} \wedge \boldsymbol{x}, \tag{51}$$
$$B(\boldsymbol{\omega})\boldsymbol{x} = -B(\boldsymbol{x})\boldsymbol{\omega}, \tag{52}$$
$$B(\boldsymbol{\omega})^2 = \boldsymbol{\omega} \otimes \boldsymbol{\omega} - |\boldsymbol{\omega}|^2 I, \tag{53}$$
$$B(\boldsymbol{\omega})^3 = -|\boldsymbol{\omega}|^2 B(\boldsymbol{\omega}), \tag{54}$$
$$RB(\boldsymbol{\omega})R^T = B(R\boldsymbol{\omega}), \tag{55}$$
$$\operatorname{tr}(B(\boldsymbol{\omega})A) = \operatorname{tr}(AB(\boldsymbol{\omega})) = 0. \tag{56}$$

Proof: Relation (51) follows from the observation that $(\boldsymbol{\omega} \wedge \boldsymbol{x})_j = \epsilon_{jik}\omega_i x_k$, and equation (52) is an immediate consequence of (51) and $\boldsymbol{\omega} \wedge \boldsymbol{x} = -\boldsymbol{x} \wedge \boldsymbol{\omega}$. To show (53), we note that $\epsilon_{njl}\epsilon_{nki} = \delta_{jk}\delta_{li} - \delta_{ij}\delta_{lk}$, and hence

$$\begin{aligned} B(\boldsymbol{\omega})^2_{ij} = B(\boldsymbol{\omega})_{in} B(\boldsymbol{\omega})_{nj} = \omega_k \omega_l \epsilon_{kni} \epsilon_{ljn} &= \omega_k \omega_l (\delta_{jk}\delta_{li} - \delta_{ij}\delta_{lk}) \\ &= \omega_i \omega_j - \omega_k \omega_k \delta_{ij} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega} - |\boldsymbol{\omega}|^2 I)_{ij}. \end{aligned}$$

By multiplying (53) with $B(\boldsymbol{\omega})$ and noting that $B(\boldsymbol{\omega})\boldsymbol{\omega} = \boldsymbol{\omega} \wedge \boldsymbol{\omega} = 0$, relation (54) follows. Due to the fact that $RB(\boldsymbol{\omega})R^T$ in (55) is skew symmetric, there exists some $\bar{\boldsymbol{\omega}} \in \mathbb{R}^3$ such that $B(\bar{\boldsymbol{\omega}}) = RB(\boldsymbol{\omega})R^T$. Raising this equality to the third power and using (54), we conclude $|\bar{\boldsymbol{\omega}}|^2 B(\bar{\boldsymbol{\omega}}) = |\boldsymbol{\omega}|^2 B(\bar{\boldsymbol{\omega}})$. If we skip the trivial case $\boldsymbol{\omega} = 0$, we can assume that both $\boldsymbol{\omega}$ and $\bar{\boldsymbol{\omega}}$ are non-zero and the previous relation allows us to conclude $|\bar{\boldsymbol{\omega}}| = |\boldsymbol{\omega}|$. If we now square the relation $B(\bar{\boldsymbol{\omega}}) = RB(\boldsymbol{\omega})R^T$ and use (53), we obtain $\bar{\boldsymbol{\omega}} \otimes \bar{\boldsymbol{\omega}} = R\boldsymbol{\omega} \otimes \boldsymbol{\omega} R^T$ so that after applying a general vector $\boldsymbol{x}$ and scalar multiplying with $R\boldsymbol{\omega}$

$$(\bar{\boldsymbol{\omega}} \cdot \boldsymbol{x})(\bar{\boldsymbol{\omega}} \cdot R\boldsymbol{\omega}) = (\boldsymbol{\omega} \cdot R^T \boldsymbol{x})|\boldsymbol{\omega}|^2, \qquad \forall \boldsymbol{x} \in \mathbb{R}^3.$$

Thus, we conclude that

$$R\boldsymbol{\omega} = \frac{\bar{\boldsymbol{\omega}} \cdot R\boldsymbol{\omega}}{|\boldsymbol{\omega}|^2} \bar{\boldsymbol{\omega}}.$$

Taking norms on both sides and using the earlier result $|\boldsymbol{\omega}| = |\bar{\boldsymbol{\omega}}|$, we see that the scalar factor $(\bar{\boldsymbol{\omega}} \cdot R\boldsymbol{\omega})/|\boldsymbol{\omega}|^2$ can only be plus or minus one. Since $\bar{\boldsymbol{\omega}}$ depends quadratically on R through the relation $B(\bar{\boldsymbol{\omega}}) = RB(\boldsymbol{\omega})R^T$, the factor is a cubic polynomial (and thus continuous) in the coefficients of R. Due to the fact that $SO(3)$ is connected and that the factor equals plus one for $R = I \in SO(3)$, we have proved $\bar{\boldsymbol{\omega}} = R\boldsymbol{\omega}$. Finally, relation (56) follows from

$$\operatorname{tr}(\epsilon_i A) = \epsilon_{ijk} A_{jk} = (\epsilon_{ijk} - \epsilon_{ikj}) A_{jk}/2 = (\epsilon_{ijk} A_{jk} - \epsilon_{ikj} A_{kj})/2 = 0.$$

Note that

$$\operatorname{tr}(A\epsilon_i) = \operatorname{tr}((A\epsilon_i)^T) = -\operatorname{tr}(\epsilon_i A) = 0.$$

■

A.2 Rotations and skew symmetry

The following characterization shows that $SO(3)$ is the image of the linear space of skew symmetric matrices under the exponential map.

Lemma 2 *Let $R \in SO(3)$. Then there exists $\boldsymbol{\omega} \in \mathbb{R}^3$ such that $R = \exp(B(\boldsymbol{\omega}))$ Conversely, $\exp(B(\boldsymbol{\omega})) \in SO(3)$ for all $\boldsymbol{\omega} \in \mathbb{R}^3$.*

Proof: A simple argument shows that R has a normalized eigenvector $\boldsymbol{a}$ with corresponding eigenvalue $\lambda = 1$. In fact, R has at least one real eigenvalue (as any 3×3 matrix), $\lambda^2 = (\lambda \boldsymbol{a}) \cdot (\lambda \boldsymbol{a}) = (R\boldsymbol{a}) \cdot (R\boldsymbol{a}) = |\boldsymbol{a}|^2 = 1$, and $\det R = 1$ excludes the case that -1 can be an eigenvalue with odd multiplicity. Choosing a normalized vector $\boldsymbol{e}$ orthogonal to $\boldsymbol{a}$, we define a rotation matrix M by specifying the columns $(\boldsymbol{a}\,, \boldsymbol{e}\,, \boldsymbol{a} \wedge \boldsymbol{e})$. Then, using the skew symmetry of $B(\boldsymbol{a})$, (53) and

$$R(\boldsymbol{a} \wedge \boldsymbol{e}) = RB(\boldsymbol{a})\boldsymbol{e} = B(R\boldsymbol{a})R\boldsymbol{e} = \boldsymbol{a} \wedge (R\boldsymbol{e}),$$

we calculate

$$M^T R M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \boldsymbol{e} \cdot (R\boldsymbol{e}) & -(R\boldsymbol{e}) \cdot (\boldsymbol{a} \wedge \boldsymbol{e}) \\ 0 & (R\boldsymbol{e}) \cdot (\boldsymbol{a} \wedge \boldsymbol{e}) & \boldsymbol{e} \cdot (R\boldsymbol{e}) \end{pmatrix}$$

Since $(R\boldsymbol{e})\cdot\boldsymbol{a} = (R\boldsymbol{e})\cdot(R\boldsymbol{a}) = \boldsymbol{e}\cdot\boldsymbol{a} = 0$, we have

$$1 = |R\boldsymbol{e}|^2 = ((R\boldsymbol{e})\cdot\boldsymbol{e})^2+((R\boldsymbol{e})\cdot\boldsymbol{a})^2+((R\boldsymbol{e})\cdot(\boldsymbol{a}\wedge\boldsymbol{e}))^2 = ((R\boldsymbol{e})\cdot\boldsymbol{e})^2+((R\boldsymbol{e})\cdot(\boldsymbol{a}\wedge\boldsymbol{e}))^2$$

which implies the existence of $\varphi \in [-\pi,\pi)$ with

$$(R\boldsymbol{e})\cdot\boldsymbol{e} = \cos\varphi, \qquad (R\boldsymbol{e})\cdot(\boldsymbol{a}\wedge\boldsymbol{e}) = \sin\varphi.$$

Using the relation

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \exp\begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix}$$

we conclude $M^TRM = \exp(B(\varphi\boldsymbol{e}_1))$, respectively

$$R = \exp(MB(\varphi\boldsymbol{e}_1)M^T) = \exp(B(\varphi M\boldsymbol{e}_1)) = \exp(B(\boldsymbol{\omega}))$$

with $\boldsymbol{\omega} = \varphi\boldsymbol{a}$. The second statement follows from the observation

$$\exp(B(\boldsymbol{\omega}))\exp(B(\boldsymbol{\omega}))^T = \exp(B(\boldsymbol{\omega}) + B(\boldsymbol{\omega})^T) = \exp(0) = I,$$

and from $\det\exp(B(\boldsymbol{\omega})) = \exp(\mathrm{tr}(B(\boldsymbol{\omega}))) = \exp(0) = 1$. ■

A similar characterization for smooth families of rotation matrices is the following.

Lemma 3 *Let $R \in C^1([0,t_{max}], SO(3))$. Then there exists a function $\boldsymbol{\omega} \in C^0([0,t_{max}], R^3)$ such that*

$$\dot{R}(t) = B(\boldsymbol{\omega}(t))R(t), \qquad R(0) \in SO(3). \tag{57}$$

Conversely, if $\boldsymbol{\omega} \in C^0([0,t_{max}], R^3)$, then the solution of (57) *gives rise to a function $R \in C^1([0,t_{max}], SO(3))$.*

Proof: Define $A(t) = \dot{R}(t)R^T(t)$. Taking the derivative of $R(t)R^T(t) = I$ leads to $A(t) + A^T(t) = 0$ with $A \in C^0([0,t_{max}], R^{3\times3})$. Setting $\omega_1 = A_{32}$, $\omega_2 = A_{13}$, $\omega_3 = A_{21}$, we have shown (57). Conversely, the unique solution $R \in C^1([0,t_{max}], R^{3\times3})$ of (57) satisfies $R(t)R^T(t) = I$ because the relation holds for $t = 0$ and the time derivative vanishes. The determinant $D(t)$ of $R(t)$ follows the equation $\dot{D}(t) = \mathrm{tr}(B(\boldsymbol{\omega}(t)))D(t)$ so that $R(t) \in SO(3)$ since $D(0) = 1$. ■

In view of Lemma 2, the semigroup generated by a vector field $\boldsymbol{w}(\boldsymbol{x}) = B(\boldsymbol{\omega})\boldsymbol{x}$ is the family of rotations $\exp(tB(\boldsymbol{\omega}))$. Similarly, the vector field $\boldsymbol{w}(\boldsymbol{x}) = B(\boldsymbol{\omega})\boldsymbol{x} + \boldsymbol{v}$ generates a rigid body motion consisting of a rotation and a translation. The following Lemma shows that these vector fields are characterized by the equation $S[\boldsymbol{w}] = 0$ where $S[\boldsymbol{w}] = (\nabla\boldsymbol{w} + (\nabla\boldsymbol{w})^T)/2$ is the symmetric part of the gradient $\nabla\boldsymbol{w}$. Note that the skew symmetric part $(\nabla\boldsymbol{w} - (\nabla\boldsymbol{w})^T)/2$ of a general vector field $\boldsymbol{w}$ can be written as $B(\boldsymbol{\alpha}(\boldsymbol{x}))$ with

$$\boldsymbol{\alpha} = \frac{1}{2}\begin{pmatrix} \partial_{x_2}w_3 - \partial_{x_3}w_2 \\ \partial_{x_3}w_1 - \partial_{x_1}w_3 \\ \partial_{x_1}w_2 - \partial_{x_2}w_1 \end{pmatrix} = \frac{1}{2}\mathrm{curl}\,\boldsymbol{w}$$

so that

$$\nabla \boldsymbol{w} = \frac{1}{2} B(\operatorname{curl} \boldsymbol{w}) + S[\boldsymbol{w}]. \tag{58}$$

Lemma 4 *Let $\Omega \subset \mathbb{R}^3$ be open and connected and assume $\boldsymbol{w} \in C^2(\Omega, \mathbb{R}^3)$ satisfies $S[\boldsymbol{w}] = 0$ in Ω. Then $\boldsymbol{w}(\boldsymbol{x}) = B(\boldsymbol{\omega})\boldsymbol{x} + \boldsymbol{v}$ for some $\boldsymbol{\omega}, \boldsymbol{v} \in \mathbb{R}^3$.*

Proof: Splitting the gradient of $\boldsymbol{w}$ in its symmetric and skew symmetric parts, we have $\nabla \boldsymbol{w} = B(\boldsymbol{\alpha}(\boldsymbol{x}))$ for some function $\boldsymbol{\alpha} \in C^1(\Omega, \mathbb{R}^3)$. Since second derivatives of the components w_i commute, i.e.

$$\frac{\partial^2 w_i}{\partial x_j \partial x_k} = \frac{\partial^2 w_i}{\partial x_k \partial x_j}, \qquad i, j, k = 1, 2, 3$$

we obtain nine conditions on the unknown function $\boldsymbol{\alpha}$. Due to the zero diagonal elements of $B(\boldsymbol{\alpha})$, we immediately find (see (50))

$$\frac{\partial \alpha_3}{\partial x_1} = \frac{\partial \alpha_2}{\partial x_1} = \frac{\partial \alpha_3}{\partial x_2} = \frac{\partial \alpha_1}{\partial x_2} = \frac{\partial \alpha_2}{\partial x_3} = \frac{\partial \alpha_1}{\partial x_3} = 0.$$

The remaining three conditions are

$$-\frac{\partial \alpha_3}{\partial x_3} = \frac{\partial \alpha_2}{\partial x_2}, \quad \frac{\partial \alpha_3}{\partial x_3} = -\frac{\partial \alpha_1}{\partial x_1}, \quad -\frac{\partial \alpha_2}{\partial x_2} = \frac{\partial \alpha_1}{\partial x_1}$$

from which we conclude that all components of $\boldsymbol{\alpha}$ are constant, i.e. $\boldsymbol{\alpha}(\boldsymbol{x}) = \boldsymbol{\omega}$ for some $\boldsymbol{\omega} \in \mathbb{R}^3$. Thus $\nabla \boldsymbol{w}(\boldsymbol{x}) = B(\boldsymbol{\omega})$ which implies $\boldsymbol{w}(\boldsymbol{x}) = B(\boldsymbol{\omega})\boldsymbol{x} + \boldsymbol{v}$ for some $\boldsymbol{v} \in \mathbb{R}^3$. ■

A.3 Parameter dependent evolutions in $SO(3)$

Let $\{M^\varepsilon \in C^1(\mathbb{R}, SO(3)) : \varepsilon > 0\}$ be a family of rotation matrix evolutions with $M^\varepsilon(0) = \bar{R}$ for all $\varepsilon > 0$. According to Lemma 3, M^ε gives rise to some function $\boldsymbol{\omega}^\varepsilon$ such that

$$\dot{M}^\varepsilon(t) = B(\boldsymbol{\omega}^\varepsilon(t)) M^\varepsilon(t), \qquad M^\varepsilon(0) = \bar{R}. \tag{59}$$

If $M^\varepsilon, \boldsymbol{\omega}^\varepsilon$ have expansions of the form

$$M^\varepsilon = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \ldots, \qquad \boldsymbol{\omega}^\varepsilon = \boldsymbol{\omega}_0 + \varepsilon \boldsymbol{\omega}_1 + \varepsilon^2 \boldsymbol{\omega}_2 + \ldots$$

the coefficients are related by

$$\dot{R}_i = \sum_{k=0}^{i} B(\boldsymbol{\omega}_k) R_{i-k}, \qquad R_0(0) = \bar{R},\ R_i(0) = 0, \quad i \geq 1 \tag{60}$$

which follows from inserting the expansions into (59) and matching orders in ε. Note that $R_0(t)$ is again a rotation matrix since $\dot{R}_0 = B(\boldsymbol{\omega}_0) R_0$, $R_0(0) = \bar{R}$ but this is not true for the higher order coefficients. For example, the first order perturbation R_1 satisfies

$$\frac{d}{dt}(R_0^T R_1) = R_0^T B(\boldsymbol{\omega}_0)^T R_1 + R_0^T (B(\boldsymbol{\omega}_0) R_1 + B(\boldsymbol{\omega}_1) R_0)$$
$$= R_0^T B(\boldsymbol{\omega}_1) R_0 = B(R_0^T \boldsymbol{\omega}_1), \qquad (R_0^T R_1)(0) = 0.$$

Hence

$$(R_0^T R_1)(t) = \int_0^t B(R_0^T(s)\boldsymbol{\omega}_1(s))\, ds$$

is a skew symmetric matrix so that R_1 cannot be contained in $SO(3)$. But even though the truncated expansion

$$R^\varepsilon = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \cdots + \varepsilon^n R_n$$

fails to be an element of $SO(3)$, it is nevertheless close to the rotation matrix M^ε. We exploit this relation to obtain the following result.

Lemma 5 *Let $\boldsymbol{\omega}_0, \ldots, \boldsymbol{\omega}_n \in C^0([0, t_{max}], R^3)$ with $n \geq 1$ and define $R_0, \ldots, R_n$ as solutions of* (60)*. Then there exists $\bar\varepsilon > 0$ and $\bar c > 0$ such that, for $0 < \varepsilon \leq \bar\varepsilon$,*

$$R^\varepsilon = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \cdots + \varepsilon^n R_n$$

is invertible, $\|(R^\varepsilon)^{-1}\| \leq \bar c$, and

$$\|(R^\varepsilon)^{-1} - (R^\varepsilon)^T\| \leq \bar c \varepsilon^{n+1}, \qquad \|\dot R^\varepsilon - B(\boldsymbol{\omega}^\varepsilon) R^\varepsilon\| \leq \bar c \varepsilon^{n+1}$$

where $\boldsymbol{\omega}^\varepsilon = \boldsymbol{\omega}_0 + \varepsilon \boldsymbol{\omega}_1 + \cdots + \varepsilon^n \boldsymbol{\omega}_n$ and $\|\cdot\|$ is the sup-norm on $[0, t_{max}]$.

Proof: Since the solutions $R_0, \ldots, R_n$ of (60) are bounded in norm by some constant $c_1 > 0$, we have for $\varepsilon \leq 1$ that

$$\|R_0^T(\varepsilon R_1 + \varepsilon^2 R_2 + \cdots + \varepsilon^n R_n)\| \leq n c_1^2 \varepsilon.$$

Choosing $\bar\varepsilon = \min\{1, 1/(2c_1^2 n)\}$, we obtain invertibility of R^ε because $\|R_0^T R^\varepsilon - I\| \leq 1/2$ for $\varepsilon \leq \bar\varepsilon$ implies invertibility of $R_0^T R^\varepsilon$. Using von Neumann series, we also find the bound

$$\|(R_0^T R^\varepsilon)^{-1}\| \leq \frac{1}{1 - \frac{1}{2}} = 2$$

so that $\|(R^\varepsilon)^{-1}\| \leq 2c_1$. To show that the inverse of R^ε is essentially given by $(R^\varepsilon)^T$, we first note that

$$\dot R^\varepsilon = \sum_{i=0}^n \varepsilon^i \sum_{k=0}^i B(\boldsymbol{\omega}_k) R_{i-k} = B(\boldsymbol{\omega}^\varepsilon) R^\varepsilon - \sum_{i=n+1}^{2n} \varepsilon^i \sum_{k=i-n} B(\boldsymbol{\omega}_k) R_{i-k}$$

which also implies the estimate on $\dot R^\varepsilon - B(\boldsymbol{\omega}^\varepsilon) R^\varepsilon$. Using the relation, we find

$$\frac{d}{dt}((R^\varepsilon)^T R^\varepsilon - I) = -(R^\varepsilon)^T B(\boldsymbol{\omega}^\varepsilon) R^\varepsilon + (R^\varepsilon)^T B(\boldsymbol{\omega}^\varepsilon) R^\varepsilon$$
$$+ \varepsilon^{n+1} \sum_{i=0}^{n-1} \varepsilon^i \sum_{k=i+1}^n \left(R_{n+1+i-k}^T B(\boldsymbol{\omega}_k) R^\varepsilon - (R^\varepsilon)^T B(\boldsymbol{\omega}_k) R_{n+1+i-k}\right)$$

and since $((R^\varepsilon)^T R^\varepsilon)(0) = \bar{R}^T \bar{R} = I$, we have with a suitable constant c_2

$$\|(R^\varepsilon)^T R^\varepsilon - I\| \leq c_2 \varepsilon^{n+1}.$$

Consequently, $\|(R^\varepsilon)^T - (R^\varepsilon)^{-1}\| = \|((R^\varepsilon)^T R^\varepsilon - I)(R^\varepsilon)^{-1}\| \leq 2c_1 c_2 \varepsilon^{n+1}$. ∎

B Validation of the asymptotic expansion

Assuming that we have calculated the expansion coefficients as solutions of the equations given in section 3.2. Then we can set up the truncated expansions

$$\boldsymbol{c}^\varepsilon = \boldsymbol{c}_0 + \varepsilon \boldsymbol{c}_1 + \varepsilon^2 \boldsymbol{c}_2, \qquad \boldsymbol{\omega}^\varepsilon = \boldsymbol{\omega}_0 + \varepsilon \boldsymbol{\omega}_1.$$

For $R^\varepsilon = R_0 + \varepsilon R_1$, we just showed in appendix A, Lemma 5 that the inverse exists if $\varepsilon > 0$ is sufficiently small. Based on R^ε, $\boldsymbol{c}^\varepsilon$ we define $\boldsymbol{X}^\varepsilon(t, \boldsymbol{y}, \varepsilon) = \varepsilon R^\varepsilon(t)\boldsymbol{y} + \boldsymbol{c}^\varepsilon(t)$ with expansion coefficients

$$\boldsymbol{X}_0(t) = \boldsymbol{c}_0(t), \quad \boldsymbol{X}_1(t, \boldsymbol{y}) = R_0(t)\boldsymbol{y} + \boldsymbol{c}_1(t), \quad \boldsymbol{X}_2(t, \boldsymbol{y}) = R_1(t)\boldsymbol{y} + \boldsymbol{c}_2(t), \ldots$$

and inverse

$$\boldsymbol{Y}^\varepsilon(t, \boldsymbol{x}, \varepsilon) = \frac{1}{\varepsilon}(R^\varepsilon(t))^{-1}(\boldsymbol{x} - \boldsymbol{c}^\varepsilon(t)).$$

Finally, velocity and pressure fields are assumed to be sufficiently smooth giving rise to the truncated expansions

$$\boldsymbol{u}^\varepsilon(t, \boldsymbol{x}) = \boldsymbol{u}_0(t, \boldsymbol{x}) + \sum_{i=1}^{2} \varepsilon^i R^\varepsilon(t) \boldsymbol{u}_i(t, \boldsymbol{Y}^\varepsilon(t, \boldsymbol{x}, \varepsilon)), \tag{61}$$

$$p^\varepsilon(t, \boldsymbol{x}) = p_0(t, \boldsymbol{x}) + \sum_{i=1}^{2} \varepsilon^{i-1} p_i(t, \boldsymbol{Y}^\varepsilon(t, \boldsymbol{x}, \varepsilon)). \tag{62}$$

In the following steps, we calculate the order at which the truncated expansions satisfy the equations of section 2.4.

The rotation matrix: $\mathcal{O}(\varepsilon^2)$

For convenience, all technicalities concerning rotation matrices have been collected in the appendix A. In particular, since $R_i, \boldsymbol{\omega}_i$ satisfy equations (31), we can use Lemma 5 which shows that

$$\dot{R}^\varepsilon = B(\boldsymbol{\omega}^\varepsilon) R^\varepsilon + \mathcal{O}(\varepsilon^2),$$

i.e. R^ε is an approximate solution of (22). Moreover, Lemma 5 yields the important relations

$$(R^\varepsilon)^{-1} = (R^\varepsilon)^T + \mathcal{O}(\varepsilon^2), \qquad \|(R^\varepsilon)^{-1}\| = \mathcal{O}(1) \tag{63}$$

which will be frequently used.

The divergence condition: exact

Taking the divergence of a field

$$\boldsymbol{v}(t,\boldsymbol{x}) = R^\varepsilon(t)\bar{\boldsymbol{v}}(t,\boldsymbol{Y}^\varepsilon(t,\boldsymbol{x},\varepsilon)) \tag{64}$$

we find with Einstein's summation convention

$$\mathrm{div}\boldsymbol{v} = \frac{\partial v_i}{\partial x_i} = R^\varepsilon_{ij}\frac{\partial \bar{v}_j}{\partial y_l}\frac{1}{\varepsilon}(R^\varepsilon)^{-1}_{li} = \frac{1}{\varepsilon}\frac{\partial \bar{v}_l}{\partial y_l} = \frac{1}{\varepsilon}\mathrm{div}\bar{\boldsymbol{v}}.$$

Since all fields in (61) are divergence free, we conclude $\mathrm{div}\boldsymbol{u}^\varepsilon = 0$.

The Dirichlet condition: $\mathcal{O}(\varepsilon^3)$

We evaluate (19) at $\boldsymbol{x} = \boldsymbol{X}^\varepsilon(t,\boldsymbol{y},\varepsilon)$. Replacing terms by their approximate counterparts, we have on the right hand side

$$\boldsymbol{\omega}^\varepsilon \wedge (\boldsymbol{X}^\varepsilon - \boldsymbol{c}^\varepsilon) + \dot{\boldsymbol{c}}^\varepsilon = \varepsilon B(\boldsymbol{\omega}^\varepsilon)R^\varepsilon \boldsymbol{y} + \dot{\boldsymbol{c}}^\varepsilon = \varepsilon \dot{R}^\varepsilon \boldsymbol{y} + \dot{\boldsymbol{c}}^\varepsilon + \mathcal{O}(\varepsilon^3) = \dot{\boldsymbol{X}}^\varepsilon + \mathcal{O}(\varepsilon^3).$$

Evaluating the left hand side of (19), we need to expand the expression

$$\boldsymbol{u}_0(\cdot,\boldsymbol{X}^\varepsilon) = \boldsymbol{u}_0(\cdot,\boldsymbol{X}_0 + \varepsilon\boldsymbol{X}_1 + \varepsilon^2\boldsymbol{X}_2).$$

With the operators D_i defined in the previous section and the relation $\boldsymbol{X}_0 = \boldsymbol{c}_0$, we simply have

$$\boldsymbol{u}_0(t,\boldsymbol{X}^\varepsilon(t,\boldsymbol{y},\varepsilon)) = \sum_{i=0}^{2}\varepsilon^i (D_i(t,\boldsymbol{y})\boldsymbol{u}_0)(t,\boldsymbol{c}_0(t)) + \mathcal{O}(\varepsilon^3).$$

Hence

$$\boldsymbol{u}^\varepsilon(\cdot,\boldsymbol{X}^\varepsilon) = \boldsymbol{u}_0 + \sum_{i=1}^{2}\varepsilon^i\left(D_i\boldsymbol{u}_0 + R^\varepsilon\boldsymbol{u}_i\right) + \mathcal{O}(\varepsilon^3).$$

Inserting the expansion of R^ε and multiplying by R_0^T, we have

$$\begin{aligned}R_0^T(\boldsymbol{u}^\varepsilon - \boldsymbol{\omega}^\varepsilon \wedge (\boldsymbol{X}^\varepsilon - \boldsymbol{c}^\varepsilon) - \dot{\boldsymbol{c}}^\varepsilon) &= R_0^T(\boldsymbol{u}_0 - \dot{\boldsymbol{c}}_0)\\ &+ \varepsilon(\boldsymbol{u}_1 + R_0^T(D_1\boldsymbol{u}_0 - \dot{X}_1)) + \varepsilon^2(\boldsymbol{u}_2 + R_0^T(R_1\boldsymbol{u}_1 + D_2\boldsymbol{u}_0 - \dot{X}_2)) + \mathcal{O}(\varepsilon^3)\end{aligned}$$

which is of order ε^3 in view of (26) and (29) with $\boldsymbol{b}_i$ given by (33) and (34).

The linear momentum equation: $\mathcal{O}(\varepsilon^2)$

In order to calculate the fluid stress σ^ε corresponding to the fields $\boldsymbol{u}^\varepsilon$, p^ε, we note that the Jacobian matrix of a field $\boldsymbol{v}$ of the form (64) is

$$(\nabla\boldsymbol{v})_{ij} = \frac{\partial v_i}{\partial x_j} = R^\varepsilon_{ik}\frac{\partial \bar{v}_k}{\partial y_l}\frac{1}{\varepsilon}(R^\varepsilon)^{-1}_{lj} = \frac{1}{\varepsilon}(R^\varepsilon\nabla\bar{\boldsymbol{v}}(R^\varepsilon)^{-1})_{ij} \tag{65}$$

For $2S[\boldsymbol{v}] = \nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T$ we conclude with (63) that $S[\boldsymbol{v}] = \varepsilon^{-1}R^\varepsilon S[\bar{\boldsymbol{v}}](R^\varepsilon)^{-1} + \mathcal{O}(\varepsilon)$ so that

$$\sigma^\varepsilon = -p^\varepsilon I + 2S[\boldsymbol{u}^\varepsilon] = \sigma_0 + \sum_{i=1}^{2}\varepsilon^{i-1}R^\varepsilon\sigma_i(R^\varepsilon)^{-1} + \mathcal{O}(\varepsilon^2). \tag{66}$$

Evaluating this relation at $\boldsymbol{x} = \boldsymbol{X}^\varepsilon(t, \boldsymbol{y}, \varepsilon)$ we need to expand the σ_0 term. With the D_i notation, we obtain

$$\sigma_0(t, \boldsymbol{X}^\varepsilon(t, \boldsymbol{y}, \varepsilon)) = \sigma_0(t, \boldsymbol{c}_0(t)) + \varepsilon(D_1(t, \boldsymbol{y})\sigma_0)(t, \boldsymbol{c}_0(t)) + \mathcal{O}(\varepsilon^2).$$

In connection with (66), this yields

$$(R^\varepsilon)^{-1}\sigma^\varepsilon(\cdot, \boldsymbol{X}^\varepsilon)R^\varepsilon = (R^\varepsilon)^{-1}\sigma_0 R^\varepsilon + \sigma_1 + \varepsilon(\sigma_2 + (R^\varepsilon)^{-1}D_1\sigma_0 R^\varepsilon) + \mathcal{O}(\varepsilon^2).$$

Replacing $(R^\varepsilon)^{-1}$ by $(R^\varepsilon)^T$ on the right hand side and inserting the expansion of R^ε in the first order term, we finally obtain

$$(R^\varepsilon)^{-1}\sigma^\varepsilon(\cdot, \boldsymbol{X}^\varepsilon)R^\varepsilon = (R^\varepsilon)^T\sigma_0 R^\varepsilon + \sigma_1 + \varepsilon(\sigma_2 + R_0^T D_1\sigma_0 R_0) + \mathcal{O}(\varepsilon^2). \quad (67)$$

Noting that

$$\int_{\partial E} (R^\varepsilon)^T\sigma_0 R^\varepsilon \boldsymbol{n}\, dS = (R^\varepsilon)^T\sigma_0 R^\varepsilon \int_{\partial E} \boldsymbol{n}\, dS = 0$$

we find on the right hand side of (20) after multiplying with $|E|(R^\varepsilon)^{-1}$

$$\int_{\partial E} (R^\varepsilon)^{-1}\sigma^\varepsilon R^\varepsilon \boldsymbol{n}\, dS = \int_{\partial E} \sigma_1 \boldsymbol{n}\, dS + \varepsilon \int_{\partial E} (\sigma_2 + R_0^T D_1\sigma_0 R_0)\boldsymbol{n}\, dS + \mathcal{O}(\varepsilon^2).$$

The left hand side of (20) multiplied with $|E|(R^\varepsilon)^{-1} = |E|(R^\varepsilon)^T + \mathcal{O}(\varepsilon^2)$ yields

$$\varepsilon\varrho|E|\mathrm{Re}(R^\varepsilon)^{-1}\ddot{\boldsymbol{c}}^\varepsilon = \varepsilon\varrho|E|\mathrm{Re}R_0^T\ddot{\boldsymbol{c}}_0 + \mathcal{O}(\varepsilon^2)$$

so that, in view of (33) and (35), the linear momentum equation is satisfied up to terms of order ε^2.

The angular momentum equation: $\mathcal{O}(\varepsilon^2)$

Proceeding exactly as in the previous case, we multiply (21) by $|E|(R^\varepsilon)^{-1}$ and insert our approximate quantities. On the right hand side we find with (67) and (63)

$$\int_{\partial E} \boldsymbol{y} \wedge (R^\varepsilon)^T\sigma^\varepsilon R^\varepsilon \boldsymbol{n}\, dS = \int_{\partial E} \boldsymbol{y} \wedge (R^\varepsilon)^T\sigma_0 R^\varepsilon \boldsymbol{n}\, dS$$
$$+ \int_{\partial E} \boldsymbol{y} \wedge \sigma_1 \boldsymbol{n}\, dS + \varepsilon \int_{\partial E} \boldsymbol{y} \wedge (\sigma_2 + R_0^T D_1\sigma_0 R_0)\boldsymbol{n}\, dS + \mathcal{O}(\varepsilon^2). \quad (68)$$

Noting that the symmetric matrix $A = (R^\varepsilon)^T\sigma_0 R^\varepsilon$ is independent of the integration variable, that

$$\boldsymbol{e}_i \cdot (\boldsymbol{y} \wedge A\boldsymbol{n}) = \boldsymbol{e}_i \cdot B(\boldsymbol{y})A\boldsymbol{n} = -B(\boldsymbol{y})\boldsymbol{e}_i \cdot A\boldsymbol{n}$$
$$= B(\boldsymbol{e}_i)\boldsymbol{y} \cdot A\boldsymbol{n} = AB(\boldsymbol{e}_i)\boldsymbol{y} \cdot \boldsymbol{n} = AB(\boldsymbol{e}_i) : \boldsymbol{y} \otimes \boldsymbol{n}$$

and that, with the divergence theorem,

$$\int_{\partial E} y_k n_l\, dS = \int_E \mathrm{div}(y_k \boldsymbol{e}_l)\, d\boldsymbol{y} = |E|\delta_{kl},$$

we conclude

$$\int_{\partial E} \boldsymbol{y} \wedge A\boldsymbol{n}\, dS = |E|AB(\boldsymbol{e}_i) : I = |E|\mathrm{tr}(AB(\boldsymbol{e}_i)).$$

Using (56) in Lemma 1, appendix A, we conclude that the first integral on the right of (68) vanishes. Since the left hand side of (21) is already of order ε^2, we see that (21) is satisfied up to order ε^2 if (33) and (36) hold.

The Navier Stokes equation: $\mathcal{O}(\varepsilon)$

With the abbreviation $\boldsymbol{u}_{loc}, p_{loc}$ for the sums of local fields in (61), (62), we first characterize the different terms which appear when inserting $\boldsymbol{u}^\varepsilon = \boldsymbol{u}_0 + \boldsymbol{u}_{loc}$, $p^\varepsilon = p_0 + p_{loc}$ into the non-linear Navier Stokes equation. Since $\boldsymbol{u}_0, p_0$ satisfy (25), we find

$$\mathrm{Re}(\partial_t \boldsymbol{u}_{loc} + \boldsymbol{u}_{loc} \cdot \nabla \boldsymbol{u}_0 + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_{loc} + \boldsymbol{u}_{loc} \cdot \nabla \boldsymbol{u}_{loc}) = -\nabla p_{loc} + \Delta \boldsymbol{u}_{loc}. \quad (69)$$

As before, we evaluate this equation at $\boldsymbol{x} = \boldsymbol{X}^\varepsilon(t, \boldsymbol{y}, \varepsilon)$ and multiply both sides with $(R^\varepsilon)^{-1}$. Elementary calculations in connection with (63) show that

$$(R^\varepsilon)^{-1} \nabla p_{loc} = \sum_{i=1}^{2} \varepsilon^{i-2} (R^\varepsilon)^{-1} ((R^\varepsilon)^{-1})^T \nabla p_i = \sum_{i=1}^{2} \varepsilon^{i-2} \nabla p_i + \mathcal{O}(\varepsilon),$$
$$(R^\varepsilon)^{-1} \Delta \boldsymbol{u}_{loc} = \sum_{i=1}^{2} \varepsilon^{i-2} \frac{\partial^2 \boldsymbol{u}_i}{\partial y_l \partial y_k} \left[(R^\varepsilon)^{-1} ((R^\varepsilon)^{-1})^T\right]_{kl} = \sum_{i=1}^{2} \varepsilon^{i-2} \Delta \boldsymbol{u}_i + \mathcal{O}(\varepsilon)$$

where terms on the left are evaluated at $(t, \boldsymbol{X}^\varepsilon(t, \boldsymbol{y}, \varepsilon))$ and on the right at $(t, \boldsymbol{y})$. Using (27), we conclude

$$(R^\varepsilon)^{-1}(-\nabla p_{loc} + \Delta \boldsymbol{u}_{loc}) = \sum_{i=1}^{2} \varepsilon^{i-2} (-\nabla p_i + \Delta \boldsymbol{u}_i) + \mathcal{O}(\varepsilon) = \mathcal{O}(\varepsilon). \quad (70)$$

To evaluate the time derivative $\partial_t \boldsymbol{u}_{loc}$, we first study a typical term (64). We have

$$\partial_t \boldsymbol{v}(t, \boldsymbol{x}) = \dot{R}^\varepsilon(t) \bar{\boldsymbol{v}}(t, \boldsymbol{Y}^\varepsilon(t, \boldsymbol{x}, \varepsilon)) + R^\varepsilon(t)(\partial_t \bar{\boldsymbol{v}})(t, \boldsymbol{Y}^\varepsilon(t, \boldsymbol{x}, \varepsilon))$$
$$+ R^\varepsilon(t)(\nabla \bar{\boldsymbol{v}})(t, \boldsymbol{Y}^\varepsilon(t, \boldsymbol{x}, \varepsilon)) \dot{\boldsymbol{Y}}^\varepsilon(t, \boldsymbol{x}, \varepsilon).$$

Taking space and time derivatives of the relation $\boldsymbol{Y}^\varepsilon(t, \boldsymbol{X}^\varepsilon(t, \boldsymbol{y}, \varepsilon), \varepsilon) = \boldsymbol{y}$, we obtain

$$\dot{\boldsymbol{Y}}^\varepsilon(t, \boldsymbol{X}^\varepsilon(t, \boldsymbol{y}, \varepsilon), \varepsilon) = -(\nabla \boldsymbol{X}^\varepsilon)^{-1}(t, \boldsymbol{y}, \varepsilon) \dot{\boldsymbol{X}}^\varepsilon(t, \boldsymbol{y}, \varepsilon) = -\frac{1}{\varepsilon}(R^\varepsilon(t))^{-1} \dot{\boldsymbol{X}}^\varepsilon(t, \boldsymbol{y}, \varepsilon).$$

Thus,

$$(R^\varepsilon)^{-1} \partial_t \boldsymbol{u}_{loc} = \sum_{i=1}^{2} \varepsilon^i (\partial_t \boldsymbol{u}_i + (R^\varepsilon)^T R^\varepsilon \boldsymbol{u}_i) - \sum_{i=1}^{2} \varepsilon^{i-1} \nabla \boldsymbol{u}_i (R^\varepsilon)^T \dot{\boldsymbol{X}}^\varepsilon + \mathcal{O}(\varepsilon^2).$$

In view of (70), the truncation error on the right hand side of (69) is of order $\mathcal{O}(\varepsilon)$. Truncating the time derivative at the same order, we have

$$(R^\varepsilon)^{-1}\partial_t \boldsymbol{u}_{loc} = -\nabla \boldsymbol{u}_1 R_0^T \dot{\boldsymbol{X}}_0 + \mathcal{O}(\varepsilon). \tag{71}$$

Finally, we consider the quadratic terms in (69). Writing directional derivatives $(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}$ as matrix vector products $(\nabla\boldsymbol{w})\boldsymbol{v}$, and observing (65), we find

$$\begin{aligned}(R^\varepsilon)^{-1}(\boldsymbol{u}_{loc}\cdot\nabla\boldsymbol{u}_0 + \boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_{loc} + \boldsymbol{u}_{loc}\cdot\nabla\boldsymbol{u}_{loc}) &= \sum_{i=1}^{2}\varepsilon^i (R^\varepsilon)^T\nabla\boldsymbol{u}_0 R^\varepsilon \boldsymbol{u}_i \\ &+ \sum_{i=1}^{2}\varepsilon^{i-1}\nabla\boldsymbol{u}_i (R^\varepsilon)^T\boldsymbol{u}_0 + \sum_{i,j=1}^{2}\varepsilon^{i+j-1}\nabla\boldsymbol{u}_i\boldsymbol{u}_j + \mathcal{O}(\varepsilon^2)\end{aligned} \tag{72}$$

where all expressions in $\boldsymbol{u}_i$ are evaluated at $(t,\boldsymbol{y})$ and those in $\boldsymbol{u}_0, \boldsymbol{u}_{loc}$ at $(t, \boldsymbol{X}^\varepsilon(t,\boldsymbol{y},\varepsilon))$. The latter require additional expansion which introduces the operators D_j. Note that only the second sum in (72) contains a zero order term $\nabla\boldsymbol{u}_1 R_0^T D_0\boldsymbol{u}_0$ which can be combined with (71). Since $D_0\boldsymbol{u}_0 - \dot{\boldsymbol{X}}_0 = \boldsymbol{u}_0 - \dot{\boldsymbol{c}}_0$, equation (26) implies that this zero order contribution vanishes. Thus, the Navier-Stokes equation is satisfied up to terms of order ε.

C The Green's formula

The Green's formula is based on the rule for the divergence of a product between a stress tensor $\sigma = -pI + 2S[\boldsymbol{w}]$, where $2S[\boldsymbol{w}]_{ij} = \partial_{x_i} w_j + \partial_{x_j} w_i$, and a vector field $\boldsymbol{v}$. We have

$$\mathrm{div}(\sigma\boldsymbol{v}) = (-\nabla p + \Delta\boldsymbol{w})\cdot\boldsymbol{v} + 2S[\boldsymbol{w}] : S[\boldsymbol{v}] - p\mathrm{div}\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\mathrm{div}\boldsymbol{w}, \tag{73}$$

where $A : B = A_{ij}B_{ij}$. As a first consequence, we derive an integration by parts formula.

Lemma 6 *Let $E \subset \mathbb{R}^3$ be a bounded domain with smooth boundary ∂E. Assume further that $\boldsymbol{v}_i, q_i, \boldsymbol{f}_i$, $i = 1,2$ are smooth functions on the closure of E^c with decay property*

$$|\boldsymbol{v}_i(\boldsymbol{x})| \le \frac{c}{|\boldsymbol{x}|}, \qquad |\nabla\boldsymbol{v}_i(\boldsymbol{x})|, |q_i(\boldsymbol{x})| \le \frac{c}{|\boldsymbol{x}|^2}, \qquad |\boldsymbol{f}_i(\boldsymbol{x})| \le \frac{c}{|\boldsymbol{x}|^3}$$

for $|\boldsymbol{x}| \ge \bar{R} > 0$. If $\boldsymbol{v}_i, q_i$ are solutions of the Stokes equations

$$\mathrm{div}\sigma_i = \boldsymbol{f}_i, \quad \mathrm{div}\boldsymbol{v}_i = 0 \qquad \text{in } E^c$$

with $(\sigma_i)_{kl} = -q_i\delta_{kl} + \partial_{x_k}(u_i)_l + \partial_{x_l}(u_i)_k$, they satisfy the Green's formula

$$\int_{\partial E}\boldsymbol{v}_1\cdot(\sigma_2\boldsymbol{n})\,dS + \int_{E^c}\boldsymbol{v}_1\cdot\boldsymbol{f}_2\,d\boldsymbol{x} = \int_{\partial E}\boldsymbol{v}_2\cdot(\sigma_1\boldsymbol{n})\,dS + \int_{E^c}\boldsymbol{v}_2\cdot\boldsymbol{f}_1\,d\boldsymbol{x} \tag{74}$$

where $\boldsymbol{n}$ is the outer normal field to E.

Proof: We present a proof for completeness. More details can be found in [12]. Let B_R be a ball with radius $R > \bar{R}$ such that $E \subset B_R$. The divergence theorem implies

$$\int_{E^c \cap B_R} \operatorname{div}(\sigma_2 \boldsymbol{v}_1)\, d\boldsymbol{x} = -\int_{\partial E} (\sigma_2 \boldsymbol{v}_1) \cdot \boldsymbol{n}\, dS + \int_{\partial B_R} (\sigma_2 \boldsymbol{v}_1) \cdot \boldsymbol{n}\, dS.$$

Using (73), we find $\operatorname{div}(\sigma_2 \boldsymbol{v}_1) = 2S[\boldsymbol{v}_1] : S[\boldsymbol{v}_2] + \boldsymbol{v}_1 \cdot \boldsymbol{f}_2$, so that

$$\int_{E^c \cap B_R} 2S[\boldsymbol{v}_1] : S[\boldsymbol{v}_2]\, d\boldsymbol{x} + \int_{E^c \cap B_R} \boldsymbol{v}_1 \cdot \boldsymbol{f}_2\, d\boldsymbol{x}$$
$$= -\int_{\partial E} \boldsymbol{v}_1 \cdot (\sigma_2 \boldsymbol{n})\, dS + \int_{\partial B_R} (\sigma_2 \boldsymbol{v}_1) \cdot \boldsymbol{n}\, dS.$$

Note that symmetry of σ_2 has been used in the first integral on the right. Subtracting the corresponding equation with reversed roles of indices and using the decay properties, (74) follows. ■

With a very similar proof, we obtain the following result.

Lemma 7 *Let $E \subset \mathbb{R}^3$ be a bounded domain with smooth boundary ∂E. Assume further that $\boldsymbol{v}, q$ are smooth solutions of the Stokes equation*

$$\nabla q = \Delta \boldsymbol{v}, \quad \operatorname{div} \boldsymbol{v}_i = 0 \qquad \text{in } E^c$$

with decay property

$$|\boldsymbol{v}(\boldsymbol{x})| \leq \frac{c}{|\boldsymbol{x}|}, \qquad |\nabla \boldsymbol{v}(\boldsymbol{x})|, |q(\boldsymbol{x})| \leq \frac{c}{|\boldsymbol{x}|^2}$$

for $|\boldsymbol{x}| \geq \bar{R} > 0$. Then,

$$\int_{\partial E} \boldsymbol{v} \cdot (\sigma \boldsymbol{n})\, dS = 2 \int_{E^c} S[\boldsymbol{v}] : S[\boldsymbol{v}]\, d\boldsymbol{x} \tag{75}$$

where $\boldsymbol{n}$ is the outer normal field to E and $\sigma = -qI + 2S[\boldsymbol{v}]$.

Using Lemma 7 we can show the invertibility of the matrix

$$L_{kj} = \int_{\partial E} \boldsymbol{w}_j \cdot (\sigma[\boldsymbol{w}_k]\boldsymbol{n})\, dS, \qquad k, j = 1, \ldots, 6 \tag{76}$$

where $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_6$ are of the Stokes equation in E^c without source term and with Dirichlet boundary conditions

$$\boldsymbol{w}_k = \boldsymbol{e}_k \quad k = 1, 2, 3, \qquad \boldsymbol{w}_k = \boldsymbol{y} \wedge \boldsymbol{e}_k \quad k = 4, 5, 6, \qquad \text{on } \partial E \tag{77}$$

The vectors $\boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6$ as synonym for the standard unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ in $\mathbb{R}^3$. The stress tensors corresponding to $\boldsymbol{w}_k$ are denoted $\sigma[\boldsymbol{w}_k] = -q_k I + 2S[\boldsymbol{w}_k]$ with pressure functions q_k.

Lemma 8 *Let $E \subset \mathbb{R}^3$ be a bounded domain with smooth boundary ∂E. Assume further that $\boldsymbol{w}_k, q_k$, $k = 1, \ldots, 6$ are smooth solutions of the homogeneous Stokes equation with Dirichlet conditions* (77) *on the closure of E^c with decay property*

$$|\boldsymbol{v}_k(\boldsymbol{x})| \leq \frac{c}{|\boldsymbol{x}|}, \qquad |\nabla \boldsymbol{v}_k(\boldsymbol{x})|, |q_k(\boldsymbol{x})| \leq \frac{c}{|\boldsymbol{x}|^2}, \qquad |\boldsymbol{x}| \geq \bar{R} > 0.$$

Then the matrix $L \in \mathbb{R}^{6\times 6}$ given in (76) *is invertible.*

Proof: Let us assume that $\boldsymbol{\lambda} \in \mathbb{R}^6$ is in the kernel of L. In particular, we have

$$0 = \sum_{j,k=1}^{6} \lambda_k L_{kj} \lambda_j = \int_{\partial E} \boldsymbol{w} \cdot (\sigma[\boldsymbol{w}]\boldsymbol{n})\, dS, \qquad \boldsymbol{w} = \sum_{k=1}^{6} \lambda_k \boldsymbol{w}_k.$$

According to Lemma 7, we conclude $S[\boldsymbol{w}] = 0$ and Lemma 4 implies $\boldsymbol{w}(\boldsymbol{x}) = B(\boldsymbol{\alpha})\boldsymbol{x} + \boldsymbol{v}$. Since all $\boldsymbol{w}_k$ are decaying at infinity, it follows that $\boldsymbol{\alpha} = \boldsymbol{v} = 0$ so that $\boldsymbol{w}$ vanishes. In particular, $\boldsymbol{w}$ vanishes at the boundary ∂E and since the boundary values (77) are linearly independent, all λ_k must be zero which shows the injectivity of L. Since L is a square matrix, invertibility follows. ■

D Details for the ellipsoidal geometry

D.1 The surface measure

We consider the ellipsoid

$$E = DB_1, \qquad D = \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix}, \quad d_1 > 0.$$

where $B_1 = \{\boldsymbol{z} \in \mathbb{R}^3 : |\boldsymbol{z}| < 1\}$ denotes the unit ball in $\mathbb{R}^3$. The tangent space at $\boldsymbol{y} \in \partial E$ is spanned by $D\boldsymbol{t}_1, D\boldsymbol{t}_2$ where $\boldsymbol{t}_1$, $\boldsymbol{t}_2$ are tangential at $\boldsymbol{z} = D^{-1}\boldsymbol{y}$. In particular,

$$0 = \boldsymbol{z} \cdot \boldsymbol{t}_i = (D^{-1}\boldsymbol{y}) \cdot \boldsymbol{t}_i = (D^{-2}\boldsymbol{y}) \cdot (D\boldsymbol{t}_i) \quad i = 1, 2$$

so that $\boldsymbol{n} = D^{-2}\boldsymbol{y}/|D^{-2}\boldsymbol{y}|$ is the normal vector at $\boldsymbol{y} \in \partial E$. To calculate the change in surface volume when passing from ∂B_1 integrals to ∂E integrals, we consider the rectangle with sides $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ at $\boldsymbol{z} \in \partial B_1$. The image of this rectangle has sides $D\boldsymbol{t}_1$, $D\boldsymbol{t}_2$ and its area is given by

$$\begin{aligned} \det(D\boldsymbol{t}_1\,, D\boldsymbol{t}_2\,, \boldsymbol{n}) &= \frac{1}{|D^{-2}\boldsymbol{y}|} \det(D\boldsymbol{t}_1\,, D\boldsymbol{t}_2\,, D^{-2}\boldsymbol{y}) \\ &= \frac{\det D}{|D^{-2}\boldsymbol{y}|} \det(\boldsymbol{t}_1\,, \boldsymbol{t}_2\,, D^{-3}\boldsymbol{y}) \end{aligned}$$

Projecting $D^{-3}\boldsymbol{y}$ onto the normal $D^{-1}\boldsymbol{y}$ at ∂B_1, we have

$$\det(D\boldsymbol{t}_1\,,\,D\boldsymbol{t}_2\,,\,\boldsymbol{n}) = \frac{\det D}{|D^{-2}\boldsymbol{y}|}(D^{-3}\boldsymbol{y})\cdot(D^{-1}\boldsymbol{y})\det(\boldsymbol{t}_1\,,\,\boldsymbol{t}_2\,,\,D^{-1}\boldsymbol{y}).$$

Since $\det(\boldsymbol{t}_1\,,\,\boldsymbol{t}_2\,,\,D^{-1}\boldsymbol{y})$ is the area of the original rectangle, the volume change is given by

$$dS(\boldsymbol{z}) = \frac{dS(\boldsymbol{y})}{\det D|D^{-2}\boldsymbol{y}|}, \qquad \boldsymbol{z} = D^{-1}\boldsymbol{y}. \tag{78}$$

D.2 Surface moments

Using expression (45) for the surface force on the ellipsoid

$$(\sigma[\boldsymbol{w}_k]\boldsymbol{n})(\boldsymbol{y})\,dS(\boldsymbol{y}) = \alpha_k \det D\,\boldsymbol{w}_k(D\boldsymbol{z})\,dS(\boldsymbol{z}), \qquad \boldsymbol{y} = D\boldsymbol{z} \in \partial E$$

we now calculate the required moments in (44) which are for $k = 1,\ldots,6$

$$\boldsymbol{F}_k = \int_{\partial E} \sigma[\boldsymbol{w}_k]\boldsymbol{n}\,dS, \quad \boldsymbol{M}_k = \int_{\partial E} \boldsymbol{y}\wedge\sigma[\boldsymbol{w}_k]\boldsymbol{n}\,dS, \quad S_k = \int_{\partial E} \boldsymbol{y}\otimes\sigma[\boldsymbol{w}_k]\boldsymbol{n}\,dS.$$

In general, a P moment of the surface force is given by

$$\int_{\partial E} P(\boldsymbol{y})(\sigma[\boldsymbol{w}_k]\boldsymbol{n})(\boldsymbol{y})\,dS(\boldsymbol{y}) = \alpha_k \det D \int_{\partial B_1} P(D\boldsymbol{z})\boldsymbol{w}_k(D\boldsymbol{z})\,dS(\boldsymbol{z}).$$

If P is a homogeneous polynomial, then we can write the surface integral over ∂B_1 as volume integral over B_1. In fact if Q is a homogeneous polynomial of degree l

$$\begin{aligned}\int_{B_1} Q(\boldsymbol{x})\,d\boldsymbol{x} &= \int_0^1 \int_{\partial B_1} Q(r\boldsymbol{z})\,dS\,r^2\,dr \\ &= \int_0^1 r^{l+2}\,dr \int_{\partial B_1} Q(\boldsymbol{z})\,dS = \frac{1}{l+3}\int_{\partial B_1} Q(\boldsymbol{z})\,dS\end{aligned}$$

Thus, the resulting expression for the moments of the surface force is

$$\int_{\partial E} P(\boldsymbol{y})\sigma[\boldsymbol{w}_k]\boldsymbol{n}\,dS = \alpha_k|E|(l+3)\frac{1}{|B_1|}\int_{B_1} P(D\boldsymbol{x})\boldsymbol{w}_k(D\boldsymbol{x})\,d\boldsymbol{x} \tag{79}$$

where the fields $\boldsymbol{w}_k$ inside of the body are defined as continuation of their polynomial values on the boundary

$$\boldsymbol{w}_k(\boldsymbol{y}) = \boldsymbol{e}_k \quad k = 1,2,3, \qquad \boldsymbol{w}_k(\boldsymbol{y}) = \boldsymbol{y}\wedge\boldsymbol{e}_k \quad k = 4,5,6, \qquad \boldsymbol{y}\in E$$

and l is the degree of $P\boldsymbol{w}_k$, i.e. $l = \deg P$ for $k = 1,2,3$ and $l = 1 + \deg P$ for $k = 4,5,6$.

To compute the coefficients $\boldsymbol{F}_k$ in (44), we choose $P(\boldsymbol{y}) = 1$. Noting that, because of symmetry, polynomials of odd degree average to zero over the unit ball, we immediately find

$$\boldsymbol{F}_k = \begin{cases} 3\alpha_k|E|\boldsymbol{e}_k & k = 1,2,3, \\ 0 & k = 4,5,6. \end{cases}$$

For $\boldsymbol{M}_k$ and S_k, we choose $P(\boldsymbol{y}) \in \{y_1, y_2, y_3\}$. Again, the averages of odd degree polynomials vanish which now involves $k = 1, 2, 3$. On the other hand, the product of y_i with $\boldsymbol{w}_k(\boldsymbol{y})$ gives rise to general homogeneous second order polynomials ($l = 2$). We remark that

$$\frac{1}{|B_1|}\int_{B_1} (D\boldsymbol{x}) \otimes (D\boldsymbol{x})\, d\boldsymbol{x} = D\frac{1}{|B_1|}\int_{B_1} \boldsymbol{x} \otimes \boldsymbol{x}\, d\boldsymbol{x} D = \frac{1}{5}D^2. \tag{80}$$

To calculate $\boldsymbol{M}_4, \boldsymbol{M}_5, \boldsymbol{M}_6$, we take $P(\boldsymbol{y}) = B(\boldsymbol{y})$ in (79). Since $\boldsymbol{w}_k(\boldsymbol{y}) = B(\boldsymbol{y})\boldsymbol{e}_k$, $k = 4, 5, 6$, we have $P(\boldsymbol{y})\boldsymbol{w}_k(\boldsymbol{y}) = B^2(\boldsymbol{y})\boldsymbol{e}_k = (\boldsymbol{y} \otimes \boldsymbol{y} - |\boldsymbol{y}|^2 I)\boldsymbol{e}_k$ and, in view of (79) and (80)

$$\boldsymbol{M}_k = \alpha_k |E| (D^2 - (\mathrm{tr} D^2) I)\boldsymbol{e}_k, \qquad k = 4, 5, 6.$$

Note that

$$D^2 - (\mathrm{tr} D^2) I = \frac{1}{|B_1|}\int_{B_1} (D\boldsymbol{x}) \otimes (D\boldsymbol{x}) - |D\boldsymbol{x}|^2 I\, d\boldsymbol{x} = \frac{1}{|E|}\int_E \boldsymbol{y} \otimes \boldsymbol{y} - |\boldsymbol{y}|^2 I\, d\boldsymbol{y}$$

is, up to the sign, the inertia tensor T of E defined in (23)

$$T = \begin{pmatrix} d_2^2 + d_3^2 & & \\ & d_1^2 + d_3^2 & \\ & & d_1^2 + d_2^2 \end{pmatrix}.$$

Thus,

$$\boldsymbol{M}_k = \begin{cases} 0 & k = 1, 2, 3, \\ -\alpha_k |E| T \boldsymbol{e}_k & k = 4, 5, 6. \end{cases}$$

Since the degree of $\boldsymbol{y} \otimes \boldsymbol{w}_k(\boldsymbol{y})$ is odd for $k = 1, 2, 3$, the moment matrix S_k vanishes in that case. For $k = 4, 5, 6$, we have

$$\boldsymbol{y} \otimes B(\boldsymbol{y})\boldsymbol{e}_k = -\boldsymbol{y} \otimes B(\boldsymbol{e}_k)\boldsymbol{y} = -\boldsymbol{y} \otimes \boldsymbol{y} B^T(\boldsymbol{e}_k) = \boldsymbol{y} \otimes \boldsymbol{y} B(\boldsymbol{e}_k)$$

and with $l = 2$ in (79) and (80) we eventually get

$$S_k = \begin{cases} 0 & k = 1, 2, 3, \\ \int_{\partial E} \boldsymbol{y} \otimes \sigma[\boldsymbol{w}_k]\boldsymbol{n}\, dS = \alpha_k |E| D^2 B(\boldsymbol{e}_k) & k = 4, 5, 6. \end{cases}$$

D.3 Evolution of orientation vectors

In section 4.1, we have seen that the leading order solvability conditions reduce to the equations (46)

$$-\boldsymbol{e}_k \cdot T R_0^T (\boldsymbol{\omega}_0 - \mathrm{curl}\, \boldsymbol{u}_0/2) = R_0^T S[\boldsymbol{u}_0] R_0 : D^2 B(\boldsymbol{e}_k), \quad k = 1, 2, 3$$

if the body under consideration is an ellipsoid. Using this specification of $\boldsymbol{\omega}_0$, we can set up the differential equation

$$\dot{R}_0 = B(\boldsymbol{\omega}_0) R_0, \quad R_0(0) = R(0) \tag{81}$$

for the leading order rigid body rotation R_0, or equivalently, for the three orientation vectors $\boldsymbol{p}_{0i} = R_0\boldsymbol{e}_i$.

In order to derive these equations for $\boldsymbol{p}_{0i}$, we first calculate the right hand side of (46) explicitly. Setting $A = R_0^T S[\boldsymbol{u}_0]R_0$, we find

$$A : D^2 B(\boldsymbol{e}_k) = \begin{pmatrix} (d_3^2 - d_2^2)A_{23} \\ (d_1^2 - d_3^2)A_{13} \\ (d_2^2 - d_1^2)A_{21} \end{pmatrix}_k, \qquad k = 1, 2, 3$$

and by bringing the inertia tensor T to the right hand side, (46) reduces to

$$R_0^T(\boldsymbol{\omega}_0 - \operatorname{curl} \boldsymbol{u}_0/2) = \Lambda \begin{pmatrix} A_{23} \\ A_{13} \\ A_{21} \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \frac{d_2^2-d_3^2}{d_3^2+d_2^2} & & \\ & \frac{d_3^2-d_1^2}{d_1^2+d_3^2} & \\ & & \frac{d_1^2-d_2^2}{d_2^2+d_1^2} \end{pmatrix}. \tag{82}$$

Next, we combine this result with equation (81) for which we need $B(\boldsymbol{\omega}_0)$. Applying B to the left hand side of (82) and observing (55), we find

$$B(R_0^T(\boldsymbol{\omega}_0 - \operatorname{curl} \boldsymbol{u}_0/2)) = R_0^T B(\boldsymbol{\omega}_0)R_0 - R_0^T B(\operatorname{curl} \boldsymbol{u}_0/2)R_0.$$

Denote the diagonal entries of Λ in (82) by λ_i and setting $\boldsymbol{a} = (A_{23}, A_{13}, A_{21})^T$, we obtain with symmetry of A

$$B(\Lambda \boldsymbol{a}) = \begin{pmatrix} 0 & -\lambda_3 A_{12} & \lambda_2 A_{13} \\ \lambda_3 A_{21} & 0 & -\lambda_1 A_{23} \\ -\lambda_2 A_{31} & \lambda_1 A_{32} & 0 \end{pmatrix}$$

and the differential equation for R_0 reads

$$\dot{R}_0 = B(\boldsymbol{\omega}_0)R_0 = \frac{1}{2}B(\operatorname{curl} \boldsymbol{u}_0)R_0 + R_0 B(\Lambda \boldsymbol{a}). \tag{83}$$

The matrix equation can equivalently be written as a system of equations for the columns $\boldsymbol{p}_{0i} = R_0\boldsymbol{e}_i$. Considering, for example, the first column, we find

$$R_0 B(\Lambda \boldsymbol{a})\boldsymbol{e}_1 = R_0 \begin{pmatrix} 0 & & \\ & \lambda_3 & \\ & & -\lambda_2 \end{pmatrix} A\boldsymbol{e}_1 = R_0(\lambda_3 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 - \lambda_2 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3)A\boldsymbol{e}_1.$$

Note that the tensor product terms are of the form $\epsilon_{1km}\lambda_m \boldsymbol{e}_k \otimes \boldsymbol{e}_k$. Moreover, we have $A = R_0^T S[\boldsymbol{u}_0]R_0$ and $R_0\boldsymbol{e}_k \otimes \boldsymbol{e}_k R_0^T = (R_0\boldsymbol{e}_k) \otimes (R_0\boldsymbol{e}_k)$ so that, in general,

$$R_0 B(\Lambda \boldsymbol{a})\boldsymbol{e}_i = \sum_{k,m} \epsilon_{ikm}\lambda_m \boldsymbol{p}_{0k} \otimes \boldsymbol{p}_{0k} S[\boldsymbol{u}_0]\boldsymbol{p}_{0i}, \qquad i = 1, 2, 3.$$

Hence, by applying (83) to the vectors $\boldsymbol{e}_i$, we end up with the following system of equations

$$\dot{\boldsymbol{p}}_{0i} = \frac{1}{2}\operatorname{curl} \boldsymbol{u}_0 \wedge \boldsymbol{p}_{0i} + \sum_{k,m} \epsilon_{ikm}\lambda_m \boldsymbol{p}_{0k} \otimes \boldsymbol{p}_{0k} S[\boldsymbol{u}_0]\boldsymbol{p}_{0i}, \qquad i = 1, 2, 3$$

which is supplemented by initial conditions $\boldsymbol{p}_{0i}(0) = R(0)\boldsymbol{e}_i$ according to (81).

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