ANALYSIS AND SIMULATIONS OF A REFINED FLOCKING AND SWARMING MODEL OF CUCKER-SMALE TYPE

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ABSTRACT. The Cucker-Smale model for flocking or swarming of birds or insects is generalized to scenarios where a typical bird will be subject to a) a friction force term driving it to fly at optimal speed, b) a repulsive short range force to avoid collisions, c) an attractive "flocking" force computed from the birds seen by each bird inside its vision cone, and d) a "boundary" force which will entice birds to search for and return to the flock if they find themselves at some distance from the flock. We introduce these forces in detail, discuss the required cutoffs and their implications and show that there are natural bounds in velocity space. Well-posedness of the initial value problem is discussed in spaces of measure-valued functions. We conclude with a series of numerical simulations.

1. **Introduction.** Flocking and swarming is a phenomenon widely observed in animal populations such as birds, insects, fish and even some mammals like sheep or goats. There has been significant interest in flocking among researchers with backgrounds in biology, population dynamics, ecology, engineering and applied mathematics. The advent of powerful modern computers has made it possible to replicate complex observed behaviours in computer simulations [6, 12, 15, 16, 17, 21, 22, 23]. Recently, models of Cucker-Smale type [7, 8] have attracted interest among mathematicians because of their relative simplicity, their similarity to existing models of interacting particles [19, 20], and because of their properties.

The key ingredients in a mathematical model for swarming must of course be the interaction rules between the individual animals, and it is immediately obvious that these rules will depend on the species under consideration: birds see their surroundings differently from locusts, fish eyes perceive a much different angle than sheep eyes, and so on. Realistic modelling of a swarm should take this into account. We will in this article generally talk about "birds", but it should be understood that much of our work carries over to other types of animals; the interaction rules will need adjustments from case to case.

Much as in other domains of particle dynamics (gases, plasmas, fluids, stellar systems, car traffic) there are different levels of description of animals swarms. The most basic level is that of keeping track of individual birds, leading to a system of

ordinary differential equations for the positions and velocities of each bird. This is sufficient for the modelling of a few dozen to a few thousand birds. The well-posedness of a model at this level is in principle a rather straightforward affair (although there are interesting twists when cutoff functions are involved, see Section 3); what is more interesting is the stability of the swarm behaviour as the number of birds grows very large. To this end it is useful to consider the swarm from the point of view of a kinetic model, as already done in [3, 4, 5, 9, 14]. At this kinetic level one can consider the state of the swarm as a finite measure in position-velocity space; the previous discrete level of description is included in this description as a special case, and one can investigate continuous dependence of solutions with respect to the initial state in a set of finite (and compactly supported) measures endowed with a suitable metric. Without restricting the generality, we normalize all such measures to be probability measures. A metric of choice is then the *bounded Lipschitz* or *Monge-Kantorovich-Rubinstein* distance, defined by

$$W_1(\mu, \nu) = \sup_{\|\varphi\|_{Lip} \le 1} \int \phi d(\mu - \nu), \tag{1}$$

or equivalently

$$W_1(\mu, \nu) = \inf_{\pi \in \Lambda} \int |P_1 - P_2| \, d\pi(P_1, P_2). \tag{2}$$

The integral in (1) is over $\mathbb{R}^3_x \times \mathbb{R}^3_v$, $P_i = (x_i, v_i) \in \mathbb{R}^6$, and Λ is the set of all transference plans between the measures μ and ν (see for e.g., [5, 24]). The purpose of this metric space in the context under consideration will become transparent in Section 4, where we address the well-posedness of the initial value problem for large flocks. In short, by considering measure-valued functions we are able to consider the conceptual limit of flocks with infinitely many members and show that their dynamics depend continuously on the initial state.

2. **Previous related work and critique.**

2.1. **The Cucker-Smale model.** In [4, 13, 14] the authors discussed a version of the Cucker-Smale model [8] given in discrete form as

$$\begin{cases} \dot{x}_i &= v_i \\ \dot{v}_i &= \sum_{j=1}^N m_j H(|x_i - x_j|)(v_j - v_i) \end{cases} \tag{3}$$

where $H(y) = \frac{\gamma}{(1+y^2)^\sigma}$, γ, σ are positive parameters and the m_j are masses. The system (3) is the discrete version of a nonlinear "friction" equation

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot [\xi(f)f] \tag{4}$$

with $\xi(f)(x, v, t) = \gamma \int \int \frac{v-w}{(1+|x-y|^2)^\sigma} f(y, w, t) \, dy \, dw$. The equations (3) and (4) are really the same equation; the former is a special case of the latter, arising in the case where f is a discrete measure given by $\sum m_i \delta(x - x_i(t))\delta(v - v_i(t))$.

In [13] Ha and Liu established global well-posedness of the initial value problem for (3) and (4); the solutions preserve mass *and* momentum, remain compactly supported in both position and velocity space if the data are, and the method of characteristics applies. In [4] it was shown further that the velocity support of the solutions to the kinetic equation (4) will collapse to a single point at an exponential rate, in the whole range of interaction, for $\beta \le 1/2$. This is "unconditional flocking".

This result reproduces at the continuous level those obtained in [7, 8] for the discrete model (3), and also extends partial results ($\beta \leq 1/4$ and $\beta = 1/2$) obtained in [13, 14] for the continuous model (4). While the "unconditional flocking" is a nice feature to emerge from a model, it is not what we generally see in swarms. It is not hard to see that the key ingredient in the proof of unconditional flocking is momentum conservation, and we will now discuss why this is an excessively strong constraint on the model.

2.2. **A weakness of the model.** The results from the previous references are satisfactory from a mathematical point of view, but less so from the viewpoint of a biologist. Observations (even casual observations!) suggest that while a swarm will persist for a long time, not all birds will in general acquire the same velocity. Swarms undulate, grow branches, disperse, and so on. Some of these effects can of course be attributed to interference from the outside (predators, wind, other swarms), but some are certainly intrinsic to the swarm.

The Cucker-Smale model contains in particular one feature which is unrealistic: momentum conservation, a consequence of the symmetry of the force term in (3). This is the key ingredient in the proofs of unconditional flocking given in [4, 14]. However, it is immediately clear that momentum conservation is violated even among two birds if bird 1 tries to follow bird 2, but bird 2 does not see bird 1 and hence does not react to its presence. This is one of the main issues addressed in this paper.

3. **A refined model.**

3.1. **Ingredients.** In this note we present a modified Cucker-Smale model which includes specific local rules of behaviour but does not satisfy momentum conservation. The key ideas are the following.

- Birds will make an effort to put distance between themselves and very close neighbors (short-range repulsion). We will introduce a corresponding repulsive force, independent of relative velocity. This force may or may not be included in numerical simulations - if it is neglected its only effect will be to allow for birds to "fly through" each other, or stay close together for long periods of time.
- Slow birds will conform with the above rule but will otherwise attempt to align themselves with all birds within intermediate distances; this behaviour is as depicted by the standard Cucker-Smale model.
- A bird moving with sufficiently large velocity v will react only to birds observed within a conic - shaped observation domain (with a characteristic opening angle).
- Finally, fast birds (as in the previous item) close to the edge of the swarm will correct to stay inside the swarm. This is a surface effect which will only be significant at and near the boundary of the swarm, or at the edge of "voids" inside the swarm. This effect is important to keep the swarm together.

A mathematical description of these rules requires several cut-off functions. We will generally assume that all such cut-offs are smooth; this is not a trivial assumption, as discussed later. For a finite number N of birds with equal mass m, positions $x_i(t)$ and velocities $v_i(t)$, we obtain a system

$$\dot{x}_i \;\; = \;\; v_i \tag{5}$$

$$m\dot{v}_i \;\; = \;\; R_i + A_i + B_i + (\alpha - \beta|v_i|^2)v_i \tag{6}$$

where R_i and A_i denote the repulsive and attractive parts of the interaction forces between birds, B_i is the boundary effect and the final term is a friction term (in the absence of all other forces, bird i will accelerate or slow down to a speed $\sqrt{\alpha/\beta}$). To implement our assumptions into these forces, we introduce a first (smooth) cutoff function $S_0 = S_0(r)$, i.e., $S_0(r) = 1$ for $r \le d_0$, $S_0(r) = 0$ for $r > d_0 + \epsilon_0$, and S_0 is decreasing smoothly in between (see Figure 1).
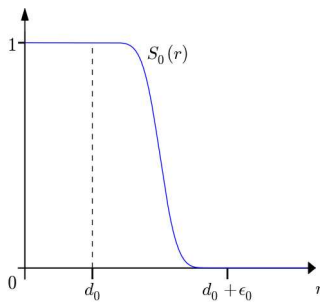

FIGURE 1. The cutoff function S_0

We set

$$R_i = \frac{\rho_1}{N} \sum_{j=1}^{N} S_0(|x_i - x_j|) \frac{x_i - x_j}{(1 + |x_i - x_j|^2)^{\beta_1}}. \tag{7}$$

Here, ρ_1 is a (large) positive parameter and $\beta_1 > 0$. There is a repulsive force between any two birds closer than d_0. One could argue that this force should become even stronger as the distance diminishes, but this is not depicted in our term - in fact, our force vanishes as $|x_j - x_i|$ approaches 0; we choose the depicted form in order to arrive at a smooth force term. In reality, birds will do their utmost to avoid collisions, but the forces they can apply to distance themselves from each other are naturally bounded. As a consequence, there may be (rare) collisions between birds, events that will be ignored here: rather, we include the theoretical possibility that birds may pass through each other, not because that is realistic, but as a mathematical convenience: a smooth interaction force permits well-posedness and stability analysis for swarms of any size.

There are several other comments in order at this point. First, since a smooth repulsive force cannot prevent collisions completely, and since we choose to ignore collisions anyway, why do we include repulsive forces at all? The answer is simple-because they are probably there in reality! Second, why don't we include force terms that really will keep birds apart, like a repulsive Coulomb-type potential? The answer to this is twofold- a) this isn't realistic at all, and b) it would introduce a serious artificial (and unneeded) difficulty into the analysis (Coulomb forces are, of course, studied in the Vlasov-Poisson and related systems and are a principal cause for the analytical challenges in this area, but there we are talking about electrons, not birds).

We return to the description of the force terms. For larger distances the repulsive forces disappear and are replaced by swarming effects, expressed in

$$A_i = \frac{1}{N} \sum_{j=1}^{N} (1 - S_0)(|x_i - x_j|)\tilde{w}(x_i - x_j, v_i)(v_j - v_i), \tag{8}$$

where the function $\tilde{w}$ is modeled after the function appearing in the standard Cucker-Smale model, with some additional cutoffs. To this end, let $S_1(v)$ be a velocity cutoff much like S_0 is a position cutoff: $S_1(|v|) = 1$ for $|v| \leq d_1$, and $S_1(|v|) = 0$ for $|v| > d_1 + \epsilon_1$ (analogous to the setup for S_0); furthermore, for large enough bird speed we introduce a "vision" cutoff: assume that a reference bird (bird 1) is at x and moves with v, where $|v| > d_1 + \epsilon_1$, and this reference bird will react only to a bird at position y (bird 2) if bird 2 is inside its observation cone, defined by $\cos \angle(y - x, v) > \delta_1$. Here, δ_1 is assumed given (but it could depend on $|v|$). We will assume further that there is a $\delta_2 < \delta_1$, such that if $\cos \angle(y - x, v) \in (\delta_2, \delta_1)$ bird 2 is seen by bird 1 peripherally (see Figure 2). Denoting $\kappa = \cos \angle(y - x, v)$ we define the cutoff S_2 on $[-1, 1]$ by $S_2(\kappa) = 0$ for $\kappa \leq \delta_2, = 1$ for $\kappa > \delta_1$, and smoothly increasing on $[\delta_2, \delta_1]$; e.g., $S_2 = 1 - S_0$.

The interpretation of this cutoff is that while the angle $\angle(y - x, v)$ increases from $\cos^{-1} \delta_1$ to $\cos^{-1} \delta_2$, bird 2 is in the range of peripheral vision of bird 1, which gradually diminishes as the angle increases and is zero once the bird is outside the cone with angle $\cos^{-1} \delta_2$.
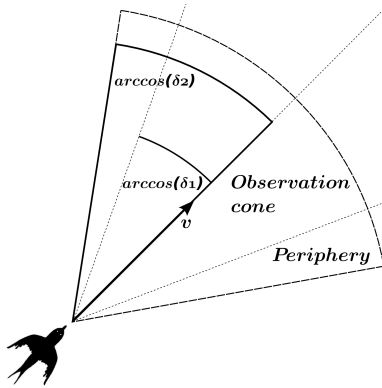

FIGURE 2. The vision cone

After these preparations we present $\tilde{w}$:

$$\tilde{w}(x - y, v) = \frac{\gamma}{(1 + |x - y|^2)^\sigma} \left\{ S_1(|v|) + (1 - S_1(|v|))S_2\left(\frac{(y - x) \cdot v}{|y - x||v|}\right) \right\}.$$

Please note that this function does indeed depend explicitly on v, in contrast to the H in the original Cucker-Smale model (3). The reason for the cutoff enforced by S_1 is that smoothness in v requires that angular dependence disappears for very low speeds. γ is a positive parameter, calibrating the "attraction" of the swarm. In fact, the need for this cutoff is largely academic: if a bird moves a very low speed at some time, then the friction term will force it to rapidly speed up and therefore leave the velocity domain where S_1 is active. It is, of course, conceivable that the

other forces (repulsion, flocking) will conspire to bring a bird towards a very low speed for a short time. This is the rare scenario when the cutoff S_1 matters.

Finally, should there be a force that prevents the swarm from breaking up? In reality, swarms do of course break up; there may be conditions which remove the rationale for swarming (whatever that may be), and the behavior of the birds will change. In the traditional Cucker-Smale model breakup of a swarm happens only very gradually or not at all, because of the symmetry in attractive forces; our main point here is that this symmetry is unrealistic, hence we lose momentum conservation, and hence breakup of a swarm is possible. Specifically, for a fast bird on the edge of the swarm and flying outward, $\tilde{w}$ will be negligibly small, so this bird will simply continue to fly outward and not change speed or direction unless overtaken by other birds.

In reality, birds "facing a void" in this way will make an effort to stay with the swarm. We will model this by requiring that birds experience a "turning" force which smoothly increases with growing "loneliness". Let us assume that we are in three dimensions and include gravity as a guiding force, pointing in the direction $-k = \langle 0, 0, -1 \rangle$. We set

$$B_i = CS_3(\rho_i)(v_i \times k), \tag{9}$$

where $\rho_i = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{1+|x_i-x_j|^2}$, C is a parameter, and S_3 is a smooth cutoff function with $S_3(\rho_i) = 1$ for $\rho_i \leq d_3$ and $S_3(\rho_i) = 0$ for $\rho_i > d_3 + \epsilon_3$. The quantity ρ_i is a possible measure of "loneliness": as ρ_i decreases, the bird will be more and more distant from other birds and will "look for the swarm". In our model it does so by flying a turn, a trend which is always active but becomes small relative to other forces if the observed "density" ρ_i is large enough. The trend to fly a turn includes a choice, and the way we have set this up assumes that the turn is flown around a vertical axis and in a clockwise direction as seen from above (the vertical axis is natural and given by gravity; the direction is arbitrary and could easily be chosen counter-clockwise. It could even vary from bird to bird, with a random variable making the choice. Here there is a lot of freedom for future numerical experiments).

Notice that the sectorial dependence which we employed for the flocking forces earlier is not used in the definition of ρ_i. One certainly has the option to do so (leading to a modification of our model), but there are arguments which can be used to justify the above ansatz. For example, birds may not only use their vision to decide that they are leaving the swarm—a bird may hear that it is at the edge of the swarm or even at some distance from the edge, and may turn around.

We summarize. Our model includes four different forces:

- Friction, which will drive each bird to fly at speed $\sqrt{\alpha/\beta}$ in the absence of other forces
- Short-range repulsive forces to avoid collisions
- Flocking forces of Cucker-Smale type, with sectorial dependence on the bird's flight direction
- Boundary effects, bringing stray birds back to the flock.

3.2. **On cutoffs.** The model described above seems unnecessarily complicated. Why the many cutoffs, and why the effort to keep them smooth? Let us briefly discuss what happens without this effort.

Consider a much simpler model, similar to the one given above, in which we completely ignore the repulsive and friction terms. As for the attractive terms, let us ignore the small velocity cutoff and assume a sharp sectorial cutoff; the simplified attraction force will read

$$\tilde{A}_i = \frac{1}{N} \sum_{j=1}^{N} \tilde{w}(x_i - x_j, v_i)(v_j - v_i), \tag{10}$$

where the function $\tilde{w}$ is modeled as before, but with simpler and discontinuous cutoffs. Assume that a reference bird (bird 1) is at x and moves with $v \neq 0$, where then this reference bird will react only to a bird at position y (bird 2) if bird 2 is inside its observation cone, defined by $\cos\angle(y-x,v) \geq \delta_1$. Here, δ_1 is assumed given (but it could depend on $|v|$). If $\cos\angle(y-x,v) < \delta_1$, bird 2 is not seen by bird 1. Denoting $\kappa = \cos\angle(y-x,v)$, we define the sharp cutoff S_4 on $[-1,1]$ (S_4 is the discontinuous version of S_2) by $S_4(\kappa) = 0$ for $\kappa \leq \delta_1$, and $= 1$ for $\kappa > \delta_1$, .

The simpler $\tilde{w}$ then reads

$$\tilde{w}(x-y,v) = \frac{\gamma}{(1+|x-y|^2)^\sigma} \left\{ S_4\left(\frac{(y-x) \cdot v}{|y-x||v|} \right) \right\}.$$

This introduces several discontinuities into the force term. First, the sectorial dependence is undefined for $|v| = 0$ (which may not matter, because it's a set of measure 0). This is the reason why we treated very slow birds separately. In practice this will not matter because birds will not stay at 0 velocity, but the mathematical problem needs attention anyway. Further, there is a discontinuity caused by the boundary of the vision cone - as the bird turns, other birds may float in and out of that cone, and the simpler attraction force based on (10) will jump every time. This is clearly a problem from both an analytical and numerical point of view, and avoided by smoothing peripheral vision, as done above. If no smoothing is done then one faces a system of ordinary differential equations with discontinuous right-hand sides, as discussed in [1, 2, 11].

As for the turning force, it is tempting to simply use

$$\tilde{B}_i = C\chi_{\{|\tilde{A}_i|<\epsilon\}} \cdot (v_i \times k),$$

with a resulting acceleration equation

$$m\dot{v}_i = \tilde{A}_i + \tilde{B}_i.$$

However, the on-off character of this turning force as a function of the birds' location is a problematic and unrealistic feature, as already mentioned earlier. One could study a model with such discontinuous force terms by using the methods introduced in the above-mentioned references, obtaining existence of solutions and numerical approximation schemes. However, in reality transitions such as peripheral vision and the drive to turn around will not happen abruptly, but according to some smooth transition law. This suggests that our mollified force terms should be more realistic than their simpler hard cutoff counterparts.

To provide an elementary example showing the issue, consider

$$\dot{x}(t) = 1 - 2\chi_{\{x(t)>1\}}, \quad x(0) = 0.$$

Trivially, $x(t) = t$ while $t < 1$, then $x(1) = 1$, and formally still $\dot{x}(1) = 1$. However, $x(t)$ cannot really exceed 1, because as soon as that would happen we would have $\dot{x}(t) = -1$, which in turn would force x below 1, and so on. This seems to suggest that $x \equiv 1$ should be a stable stationary solution, but it isn't from a formal point of view, as it does not satisfy the equation.

The problem is trivially overcome by smoothing the characteristic function $\chi_{\{x>1\}}$, and if this is done with a smooth and monotone approximation the new system will have the same overall qualitative behaviour and a stable steady solution close to 1.

3.3. **Velocity bounds.**
For the rest of the paper we will consider the model with the smoothed force laws. While it is not possible to make assertions about the asymptotic behaviour of the swarm, some things are easily seen. Let us ignore the repulsive forces (i.e., set $S_0 = 0$). We can then prove the following results on the velocity support of the swarm.

Theorem 3.1. *Let $R(t) = \max_i |v_i(t)|$ and assume that $R(0) \geq \sqrt{\frac{\alpha}{\beta}}$. Then the velocity support of the swarm stays bounded by $R(0)$.*

Proof. Since $t \to |v_i(t)|^2$ are C^1- functions, we can for each t find a time interval $[a, b]$ which contains t, and an index i_0 such that $R(\tau) = |v_{i_0}(\tau)|$ for all $\tau \in [a, b]$. We may therefore assume without restricting the generality that $R(t) = |v_1(t)|$ and that $|v_1(t)| \geq \sqrt{\frac{\alpha}{\beta}}$. Then by Cauchy's inequality, $v_1^2 - v_j \cdot v_1 \geq 0$, and therefore

$$
\begin{aligned}
\frac{d}{dt} R^2(t) &= 2v_1 \cdot \dot{v}_1 \\
&= \frac{1}{m} \left(2(\alpha - \beta|v_1|^2)v_1^2 + \frac{2}{N} \sum_{j=1}^{N} \tilde{w}(x_1 - x_j, v_1)(v_j - v_1) \cdot v_1 + 2B_1 \cdot v_1 \right) \\
&\leq 0.
\end{aligned}
$$

The last inequality holds because the first term on the right is nonpositive, the $\tilde{w}(x_1 - x_j, v_1)$ in the second term are nonnegative, and hence the second term is also nonpositive, and the last term is zero. We have in particular proved that $\dot{R}(t) \leq 0$ while $R(t) \geq \sqrt{\frac{\alpha}{\beta}}$. This completes the proof. $\qquad\square$

3.4. **The associated kinetic equation.**
As in [5] we use $f = f(x, v, t)$ to denote a measure-valued function whose value at time t is the measure $d\mu_t(x, v) = f(x, v, t)dvdx$. This notation suggests that we are only considering measures which are absolutely continuous with densities $f(\cdot, t)$; however, general finite measures are possible. We simply continue to use the above notation for convenience, and the subsequent partial differential equation is to be interpreted in the weak sense.

The measure-valued function $t \to \mu_t$ is the natural extension of the previously discussed finite model as the number of birds goes to ∞. The previous system of equations for the swarm then turns (formally, and, as shown later, rigorously) into the kinetic equation

$$\partial_t f + v \cdot \nabla_x f = -\mathrm{div}_v\Big\{ \big[\xi(f) + (\alpha - \beta|v|^2)v\big] f \Big\} \tag{11}$$

with $\xi(f) = \xi_R(f) + \xi_A(f) + \xi_B(f)$, and

$$\xi_R(f)(t,x) = \rho_1 \int\int \frac{x-y}{(1+|x-y|^2)^{\beta_1}} S_0(|x-y|) f(y,w,t) \ dw \ dy, \tag{12}$$

$$\xi_A(f)(t,x,v) = \int\int \tilde{w}(x-y,v)(w-v)(1 - S_0(|x-y|)) f(y,w,t) \ dw \ dy, \tag{13}$$

and

$$\xi_B(f)(t,x,v) = C S_3(\tilde{\rho}(x,t))(v \times \mathbf{k}). \tag{14}$$

where $\tilde{\rho}(x,t) = \int\int \frac{f(y,w,t)}{1+|x-y|^2} \ dw \ dy$

4. **On well-posedness.** For the purpose of this section the specific form of the force terms in Section 3 is not important. What matters is that the force fields $\xi_R(f)(x), \xi_A(f)(x,v)$ and $\xi_B(f)(x,v)$ satisfy uniform Lipschitz estimates on compact sets, a property which is guaranteed by the smooth cutoffs and the structure of these forces. Assume for the moment that the measure valued function $t \to \tilde{\mu}_t$ is given and uniformly (for $t \in [0,T]$) compactly supported in x,v. Abusing the notation as indicated earlier, we also just write $\tilde{f}(x,v,t)$ for this measure. We then abbreviate $E(t,x,v) = E_{\tilde{\mu}}(t,x,v) := \xi(\tilde{f})(t,x,v)$.

As we assume that $\tilde{f}$ is given, E is given, and we can compute the f which evolves according to the equation (11) by solving the characteristic system of equations

$$\frac{d}{dt}x = v \tag{15}$$

$$\frac{d}{dt}v = E(t,x,v) + (\alpha - \beta|v|^2)v. \tag{16}$$

The solvability on $[0,T]$ of the initial value problem associated with (15,16) depends, of course, on the properties of E. We list three desirable properties of E; that our definition of E actually implies these properties is stated later as a lemma.

P0. E is continuous on $[0,T] \times \mathbb{R}^6$.

P1. There is a $C > 0$ such that $\forall t \in [0,T]$

$$|E(t,x,v)| \le C(1 + |x| + |v|)$$

(sublinear growth).

P2. For each compact subset $K \subset \mathbb{R}^3 \times \mathbb{R}^3$ there is a constant L_K such that on K

$$|E(t,x,v) - E(t,y,w)| \le L_K(|x-y| + |v-w|).$$

The properties P0-P2 provide well-posedness of the initial value problem belonging to (15,16), and we define the flow $\{T_E^t\}$ associated with this system. Abbreviate $P = (x,v), P(t) := (x(t), v(t))$, then the family of transformations $\{T_E^t\}$ is given by $T_E^0 = id, T_E^t(P) = P(t)$, where $P(t)$ is the solution to (15,16) satisfying $P(0) = P$.

Assume further that μ_0 is an initial probability measure, compactly supported in x, v. If we set $\mu_t := T_E^t \# \mu_0$ (defined by $\int \int \varphi(P) \, d\mu_t(P) = \int \int \varphi(T_E^t(P)) \, d\mu_0(P)$) then $t \to \mu_t$ is a weak solution of

$$\partial_t f + v \cdot \nabla_x f = -\text{div}_v \Big\{ \big[(E + (\alpha - \beta |v|^2) v \big] f \Big\}.$$

Let $X := C([0, T], P_{1,c}(\mathbb{R}^6))$ be the set of continuous functions $t \to \mu_t$ into the compactly supported probability measures, equipped with the metric $\sup_{t \in [0,T]} W_1(\mu_t, \nu_t)$. This set is a subset of the complete metric space $C([0, T], P_1(\mathbb{R}^6))$ equipped with the same metric; here $P_1(\mathbb{R}^6)$ denotes the set of probability measures μ on $\mathbb{R}^6$ with $\int \int_{\mathbb{R}^6} (|x| + |v|) \, d\mu(x, v) < \infty$. For $\tilde{\mu}_. \in X$ we define μ_t by $\mu_t := T_{E_{\tilde{\mu}_.}}^t \# \mu_0$, where $E_{\tilde{\mu}_.} = \xi(\tilde{f}) = \xi_R(\tilde{f}) + \xi_A(\tilde{f}) + \xi_B(\tilde{f})$ is defined by (12)–(14). This defines a mapping $\Gamma : \tilde{\mu}_. \to \mu$ in X. The following properties apply.

Lemma 4.1. *The field $E_{\tilde{\mu}_.}$ satisfies properties P0, P1 and P2.*

Theorem 4.2. *The mapping Γ possesses a unique fixed point in X for some conveniently chosen $T > 0$. This fixed point is constructible via a contraction mapping argument in X and is a unique weak solution of (11).*

Theorem 4.3. *The solution given by Theorem 4.2 depends continuously on the initial measure μ_0. Specifically, there is a continuous function $r : [0, T] \to \mathbb{R}_+$ with $r(0) = 1$, such that if $\mu_.$ and $\nu_.$ are solutions of (11) with initial measures μ_0 and ν_0 respectively, then*

$$W_1(\mu_t, \nu_t) \leq r(t) W_1(\mu_0, \nu_0).$$

Details of the proofs of these results are given in [5]. They follow the blueprint from classical references such as [10], [19] and [20].

The discrete case is included in these results by simply setting $\mu_0 = \frac{1}{N} \sum \delta_{(x_{i,0}, v_{i,0})}$ and $\mu_t = \frac{1}{N} \sum \delta_{(x_i(t), v_i(t))}$. Theorem 3 above provides in particular convergence of a "particle" method, where a general μ_0 is approximated in the weak sense by a discrete measure. It follows that the behaviour of a very large flock of birds can (in principle, and on finite time intervals) be simulated by solving a system of ODEs for a reasonably large number of birds.

5. **Simulations.** Numerical simulations were implemented using MATLAB: the ordinary differential equation is solved using the MATLAB routine *ode45* which requires smooth arguments (this is one reason why smooth cutoffs are used). The smooth cutoff functions $S_0(x; d, \epsilon), S_2(x; d, \epsilon), S_3(x; d, \epsilon)$ are implemented as in [18]:

$$S_i(x; d, \epsilon) = \begin{cases} 1 & \text{if} \quad x \leq d \\ \frac{1}{2} + \frac{1}{2} \tanh \left(\frac{1}{x-d} + \frac{1}{x-(d+\epsilon)} \right) & \text{if} \quad d < x < d + \epsilon \\ 0 & \text{if} \quad x \geq d + \epsilon \end{cases}$$

for $i = 0, 3$, and $S_2(\kappa; \delta_1, \delta_2) = 1 - S_0(\kappa; \delta_2, \delta_1 - \delta_2)$.

The seven remaining figures depict initial conditions followed by end conditions for three simulation scenarios. In each picture we depict a three-dimensional projection of the birds' positions on the left and a projection of the positions from above on the right, where the arrows are the two-dimensional projections of the birds' velocities. The first scenario depicts a flock of birds (with randomly oriented

velocities) that disperse into fragments. In fact, at a later time, the fragments rejoin one another (not shown).

The second scenario depicts two flocks: a flock of birds with identical velocities approaches a flock with randomly oriented velocities. Similar to the first scenario, the randomly oriented flock splits into two components. One component joins the flock that started out with identical velocities.

Both the first and second scenarios used a total of 100 birds and the same parameters, shown in the table below. In the second scenario, the birds were divided evenly between the two initial flocks.

The third scenario (Figures 7-9) used the same parameters (but a larger number of birds - 200). Here we modified the turning behaviour of the birds such that every other bird would choose to turn in the opposite direction (left rather than right; in fact, we have observed that swarms of seagulls seem to conform with this in the sense that there are "left turners" and "right turners". Of course, birds may change from one camp to the other at some time, but this does not happen indiscriminately. For our simulations we simply assumed that there were two well defined groups for the duration of the simulation). Thus the birds are evenly divided into two groups with left and right turning preferences, respectively. The initialization was otherwise identical to the first scenario (Fig. 7). It was observed that double-milling behaviour emerged (Fig. 8) for an intermittent period. At some later time a transition occurred in which the birds' planar velocities reached a relatively equilibrated state, and the flock then experienced upwards rigid motion behaviour. One can speculate what further modifications to our model will produce yet more realistic effects, for example undulations such as observed in large flocks of starlings; we intend to carry on further simulations towards this end.

In our third simulations the two-dimensional nature of the turning force results in the upward preference shown in this rigid motion; a more general force term would enable more general rigid-body motion to occur (translations and rotations).

5.1. **Parameter Values.** The parameters used to produce the simulations are :

| Simulation Parameters			
General	N	100	Number of Birds
	$\beta_1 = \sigma$	0.5	Exponent for force terms
	d_0	1	Parameter of S_0 cutoff
	ϵ_0	1	Parameter of S_0 cutoff
Repulsion Force	ρ_1	10	Repulsive coefficient
Attraction Force	γ	10	Attractive coefficient (in $\tilde{\omega}$)
	d_1	0.5	Parameter for vision cutoff $S_1(\|v\|)$
	ϵ_1	0.5	Parameter for vision cutoff $S_1(\|v\|)$
	$\cos^{-1}\delta_1$	1.047	Parameter for angular cutoff $S_2(\kappa)$
	$\cos^{-1}\delta_2$	1.57	Parameter for angular cutoff $S_2(\kappa)$
Turning Force	C	0.003	Turning coefficient
	d_3	0.0015	Parameter for density cutoff S_3
	ϵ_3	0.0003	Parameter for density cutoff S_3
Friction Force	α	1	Friction α
	β	1	Friction β
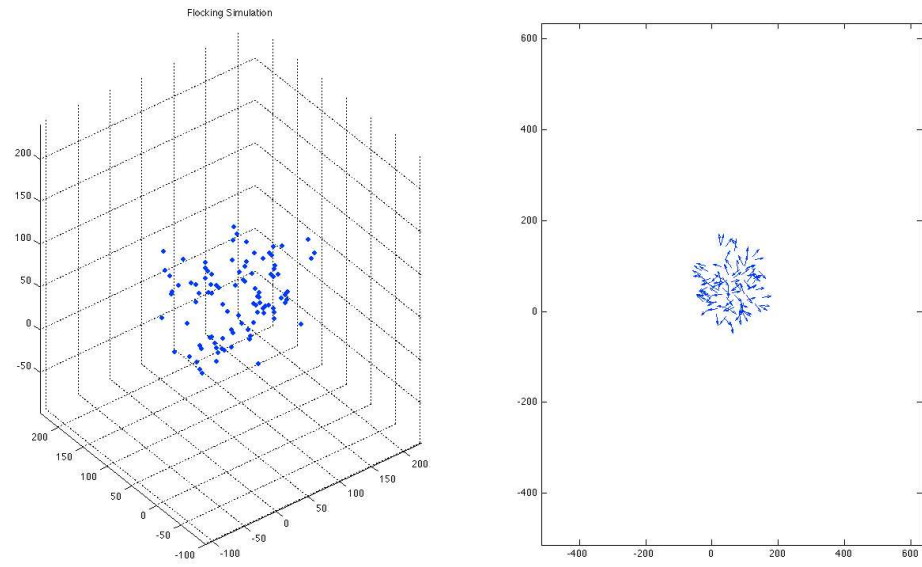
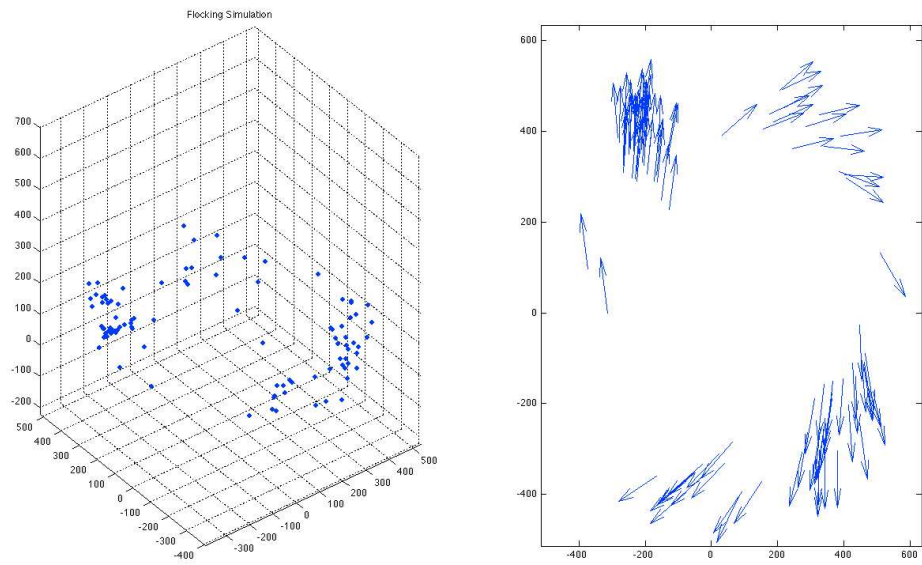FIGURE 3. Randomly distributed flock splitting (initial time)

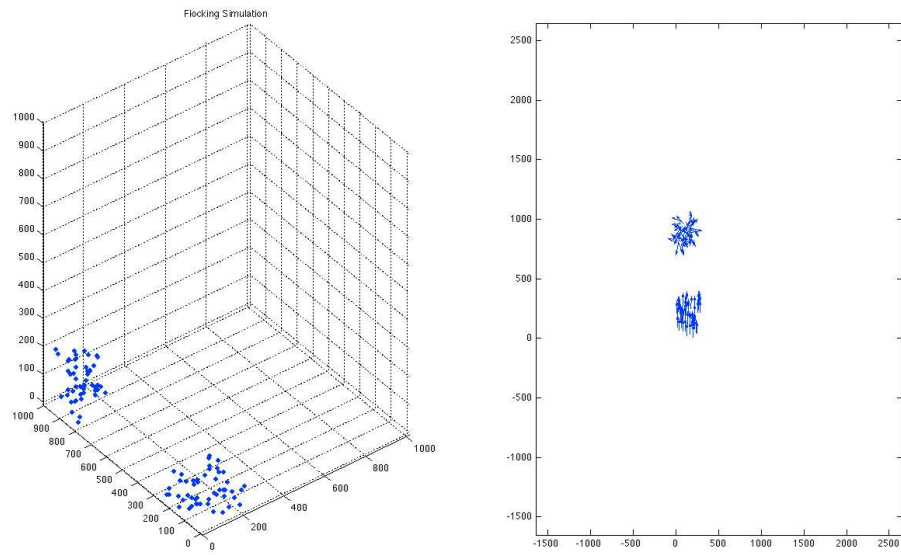FIGURE 4. Randomly distributed flock splitting (later time)

FIGURE 5. Directional flock merging with portion of random flock (initial time)
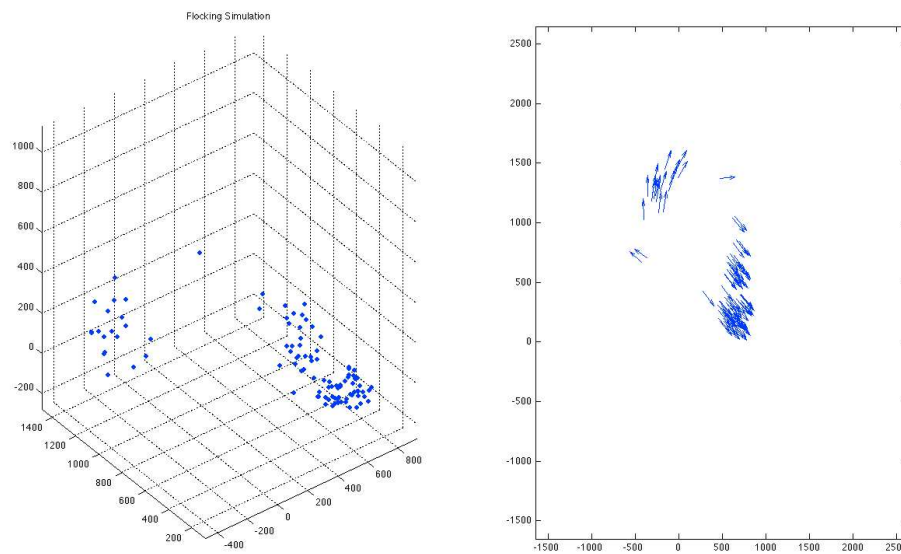

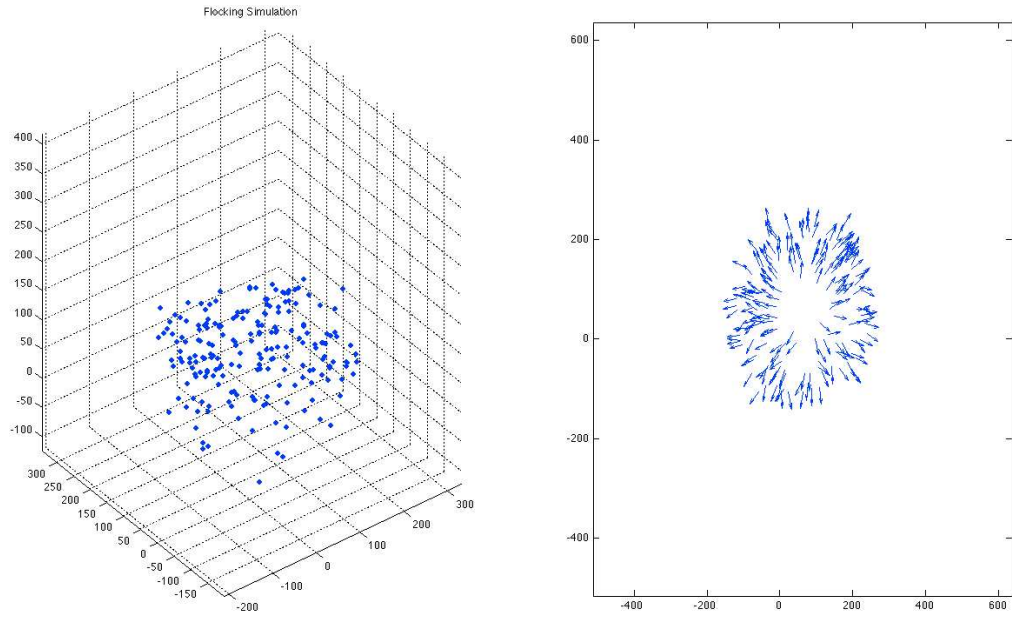FIGURE 6. Directional flock merging with portion of random flock (later time)

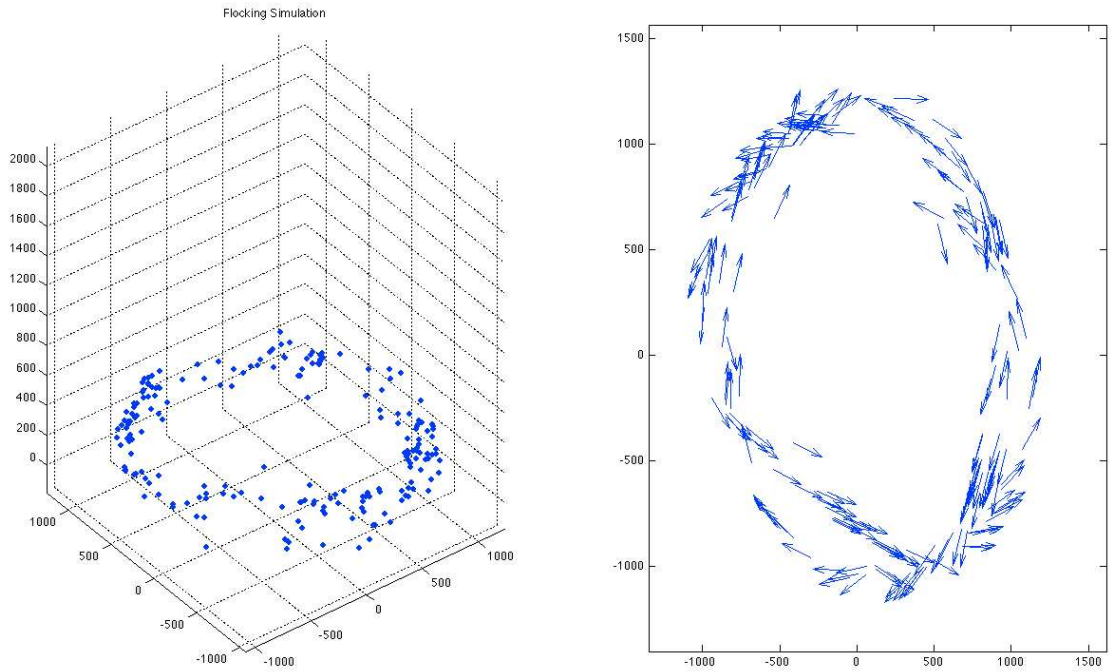FIGURE 7. Randomly distributed flock with left-right turning (initial time)

FIGURE 8. Randomly distributed flock with left-right turning (intermediate time) - double milling behaviour.
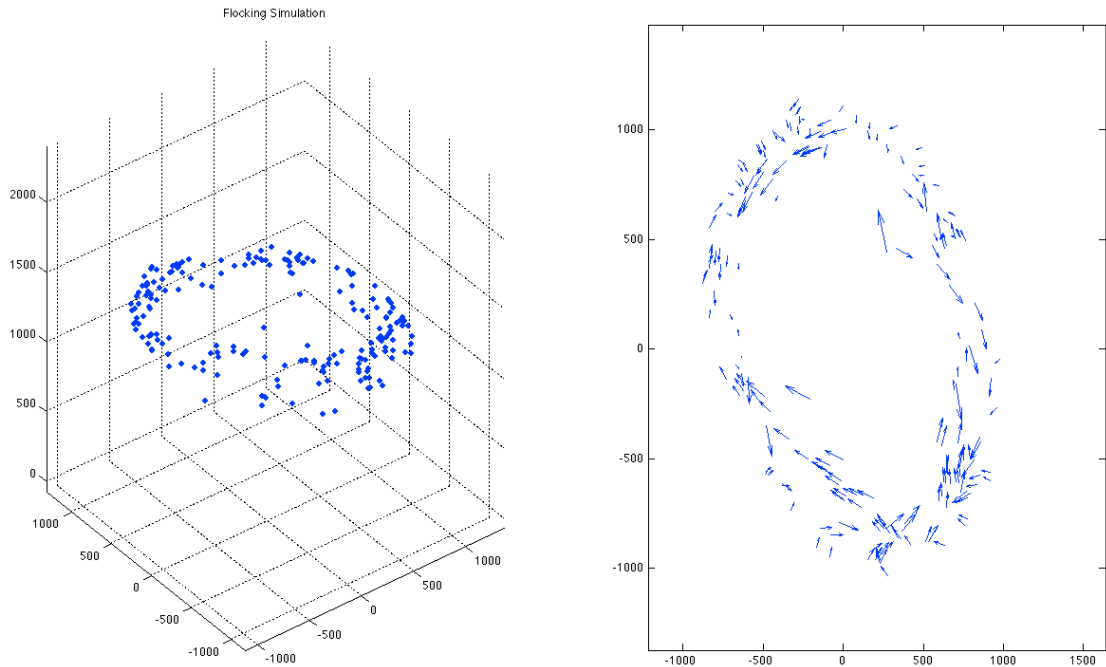
FIGURE 9. Randomly distributed flock with left-right turning (later time). Small planar velocity indicates vertical rigid motion.

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