KINETIC EQUILIBRIA IN TRAFFIC FLOW MODELS

by

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ABSTRACT

We discuss the existence and properties of nontrivial kinetic equilibria solutions for Enskog-type models of multilane traffic flow. Under certain conditions on driver behaviour it is proved that only trivial (synchronized flow) equilibria exist. For a simple explicit form of driver behaviour and a modification of the interaction terms by artificial diffusion terms, these trivial equilibria become smooth and can be computed by ODE methods. Finally, a more realistic model for driver behaviour is suggested, leading to diffusion terms which are consistent with the existence of trivial (synchronized flow) equilibria. Numerical tests reveal that the stable equilibria associated with this behaviour include bimodal equilibria for certain parameter choices, consistent with real traffic observations.

1. Introduction. In close analogy to statistical physics, there are three possible levels of description of traffic dynamics: 1) the microscopic level, where one keeps track of the velocity and position of each vehicle; 2) the macroscopic level, where the description is at the level of partial differential equations of conservation type for macroscopic variables such as density and flux; and 3) the (intermediate) kinetic level, where the position and momentum of each vehicle is ignored in favour of a distribution density $f = f(t, x, v)$. In the present study we are concerned with models of this third type and will consider examples of kinetic multilane models as suggested by Klar and Wegener in [1-2]. We mention here in passing that earlier attempts of using kinetic modeling in traffic dynamics were made by Nelson [3], Prigogine and Andrew [4], Prigogine and Herman [5], and others (see the references in [1-2]). Recent work on "fundamental diagrams" relating density and flux in macroscopic models can be found in Nelson and Sopasakis [6].

Kinetic modelling has the potential of providing the crucial ingredients towards valid simulations of real traffic. Microscopic models suffer from two drawbacks: 1) It is expensive and slow to keep track of individual cars, and 2) the statistical data obtained from such a procedure may be difficult to interpret in the absence of a higher level model, i.e., it may be hard or nearly impossible to use data from microscopic observations to predict what happens in a different but similar situation. On the opposite end of the modelling spectrum, it happens that macroscopic models which are derived without reference to microscopic or kinetic background models make unrealistic predictions in some scenarios (examples for this are given in [7]), or they may ignore some relevant dependent variables altogether.

Practical considerations make it highly desirable to derive macroscopic models which are as accurate as possible, because such models would be usable for real-time numerical simulations and provide realistic, interpretable results. Finding such macroscopic models is therefore a main goal in the modeling process. The significance of kinetic modeling is that 1) it provides a natural approach for the derivation of realistic macroscopic models, and 2) it can be used for numerical simulations directly. The approach mentioned under 1) follows in essence the transition path from rarefied gas dynamics to compressible gas dynamics, i.e., the transition from the Boltzmann equation to the compressible Euler equations in the neighborhood of (local) thermodynamic equilibrium.

The process is well known and can be summarized in a few lines as follows: A good kinetic model will give rise to an infinite set of moment equations for the infinitely many velocity moments of the density function; "good" macroscopic equations will emerge from a "good" closure relation, which may be a constitutive relation linking some higher order moments, a projection principle or an entropy maximization principle (if the system satisfies an entropic principle). Such closure methods in rarefied gas dynamics have received much attention in recent years [9].

For the traffic flow problem, the standard approach uses closure of the moment equations in the neighborhood of a kinetic equilibrium. We refer to [2] and [7] for details and simply observe that other closure procedures are hard to think of for traffic flow since there is no information about entropy functionals. In any case, this is a good reason to investigate kinetic equilibria for traffic flow: they give natural closure relations for the moment equations and therefore good macroscopic models. Of course, the kinetic equilibria can be used for other purposes in their own right, e.g., for studies of possible fundamental diagrams (linking density and flux) in equilibrium settings.

The topic of our work is therefore the existence and nature of non-trivial spatially homogeneous equilibria solutions for cumulative descriptions of multilane kinetic models. We leave the proper formulation of the problem for Section 2. Throughout the paper, an equilibrium will be called trivial if it corresponds to the situation where all cars are moving at the same fixed speed, i.e., $f(x,v) = \rho(x)\delta_{v_0}(v)$. In a spatially homogeneous context (only feasible from a statistical point of view) ρ will be a positive constant. Trivial equilibria in this sense are desirable from a practical point of view, but they are not expected to be stable for moderate to high densities, and therefore not usable for the derivation of good macroscopic models in such regimes.

2. The Problem. As in [2,7] we consider a kinetic model describing highway traffic in a cumulative picture averaging over all lanes. $f = f(t, x, v)$ is the value of the car density at time t, location $x \in \mathbb{R}$ and speed $v \in [0, 1]$ (for convenience, we assume that units have been chosen such that the speed limit is 1). The total density is $\rho(x) = \int_0^1 f(x, v)\, dv$, and we set $f(x, v) = \rho(x)F(x, v)$. F is the probability density in v of cars at x.

The models introduced in [2,7] are Enskog-type kinetic models in the sense that a driver's reaction depends on developments *ahead* of the car. To this end, let H_B and H_A be the thresholds for braking and acceleration, defined by

$$H_X(v) = H_0 + vT_X, \qquad X = A, B$$

where T_B, T_A are reaction times (it is reasonable to expect $T_B < T_A$) and H_0 is the minimal possible distance between vehicles (the average length of a car).

A driver will accelerate only if his distance to the leading car exceeds H_A, and brake only if this distance becomes smaller than H_B. It will be assumed that velocity adjustments are instantaneous, but various models will differ in assumptions on how the new velocity is chosen; as we will see, this is all-important. Following [7], we denote by $q_B(\rho), q_A(\rho)$ the correlation functions (the density-dependent probabilities of encountering a leading vehicle) for braking or accelerating. If a slower leading vehicle is present, the driver makes a choice of braking or passing (overtaking); the braking probability will be denoted by $P_B = P_B(\rho)$. This is a simplified picture of the traffic dynamics on a multilane highway where driver interactions lead to lane changes, braking, or acceleration. We refer to [2] for discussions on the nature of the correlation functions q_A and the braking probabilities P_B.

The cumulative kinetic model as introduced in [2,7,8] is

$$\partial_t f + v\partial_x f = C^+(f), \tag{2.1}$$

where the Enskog-type collision operator $C^+(f)$ is

$$C^+(f) = P_B q_B (G_B^+ - L_B^+)(f) + q_A(G_A^+ - L_A^+)(f)$$

and the various gain (G_B^+, G_A^+) and loss (L_B^+, L_A^+) terms due to braking and acceleration are

$$G_B^+(f)(x,v) = \int_0^1 \int_{v_2}^1 |v_1 - v_2| \sigma_B(v_1 \to v; v_2) f(x, v_1) F(x + H_B(v_1), v_2)\, dv_1 dv_2,$$

$$L_B^+(f)(x,v) = \int_0^v |v - v_2| f(x,v) F(x + H_B(v), v_2)\, dv_2.$$

Here $\sigma_B(v_1 \to v; v_2)$ denotes the probability density that a driver moving at speed v_1 and encountering a leading vehicle at speed $v_2 < v_1$ will brake to speed v. Similarly,

$$G_A^+(f)(x,v) = \int_0^1 \int_0^{v_2} |v_1 - v_2| \sigma_A(v_1 \to v; v_2) f(x, v_1) F(x + H_A(v_1), v_2)\, dv_1 dv_2,$$

$$L_A^+(f)(x,v) = \int_v^1 |v - v_2| f(x,v) F(x + H_A(v), v_2)\, dv_2$$

and $\sigma_A(v_1 \to v; v_2)$ denotes the probability density that a driver moving at speed v_1 and encountering a leading vehicle at speed $v_2 > v_1$ will accelerate to v.

Equilibria solutions of (2.1) are density functions $f = f(v)$ such that both sides of (2.1) vanish identically. As f is by definition independent of time and location, the defining condition for an equilibrium becomes $C(f) = 0$, where C denotes the spatially homogeneous interaction ("collision") operator.

In [7], Eqn. (2.1) is changed to

$$\partial_t f + v \partial_x f = C^+(f) + \nu(\rho)(G_s - L_s)(f), \tag{2.2}$$

with $G_s(f)(x,v) = \int_0^1 f(x,w)\, dw$ and $L_s(f)(x,v) = f(x,v)$. This correction is an artificially introduced relaxation term which accounts for "random behaviour of the drivers." It is evident that this correction is not consistent with the existence of trivial equilibria and will therefore enforce smooth equilibrium solutions (if equilibria exist at all). Indeed, for the sensible choices

$$\sigma_B(v_1 \to v; v_2) = \frac{1}{(v_1 - v_2)} \chi_{[v_2, v_1]}(v), \quad \sigma_A(v_1 \to v; v_2) = \frac{1}{(v_2 - v_1)} \chi_{[v_1, v_2]}(v), \tag{2.3}$$

and $\nu(\rho) = const.$ these equilibria are computed explicitly in [7].

In the spatially homogeneous situation, the spatial nonlocality of the collision operator disappears, and the collision operator becomes an interaction operator of Boltzmann type. This is the operator which we will study in our equilibrium search. We will drop the + superscript for any of the terms in this homogeneous situation, i.e., $C(f), G_B(f)$, etc.

The plan of the present paper is as follows. We first set $\nu(\rho) \equiv 0$ and prove in Section 3 that for a wide class of possible densities σ_A and σ_B, all equilibria are trivial. In Section 4 we generalize the model (2.2) from [7] by consideration of more universal relaxation terms and review the identification of non-trivial equilibria via the solution of ordinary differential equations. Finally, in Section 5, we present an ansatz for more realistic $\sigma_A s$ and $\sigma_B s$ which naturally entails the presence of relaxation terms in the collision operator, and we present numerical calculations of some of the corresponding equilibria. These calculations demonstrate in particular that for certain parameter regimes the emerging equilibria are bimodal, i.e., they have two separated peaks, a feature which is observed in reality.

2.1 Review of previous choices for σ_A, σ_B. Our research was inspired by the question which key features of σ_A and σ_B prevent (or allow) the existence of nontrivial equilibria. To this end, we first revisit the examples which were studied elsewhere in the literature. In [2] and [8], with suitable choices of parameters $0 \le \beta \le 1$ and $1 \le \alpha \le \infty$, the probability densities

$$\begin{aligned} \sigma_B(v_1 \to v; v_2) &= \frac{1}{(1-\beta) v_1} \chi_{[\beta v_1, v_1]}(v) \\ \sigma_A(v_1 \to v; v_2) &= \frac{1}{\min(1, \alpha v_1) - v_1} \chi_{[v_1, \min(1, \alpha v_1)]}(v) \end{aligned} \tag{2.4}$$

were used. For the analytical study in [8], the parameters α and β were set to $\beta = 0$ and $\alpha = \infty$, resulting in

$$\begin{aligned} \sigma_B(v, v_1) &= \frac{1}{v_1} \chi_{[0,v_1]}(v) \\ \sigma_A(v, v_1) &= \frac{1}{1 - v_1} \chi_{[v_1,1]}(v) \end{aligned} \tag{2.5}$$

Note that these probability densities are independent of the speed of the leading vehicle; while this is certainly an unrealistic feature of the model, it was established (by numerical experiments in [2], and in a special case with extra symmetries by an analytical argument in [8]) that there are in this case both trivial and non-trivial equilibria.

In [7], σ_A and σ_B are chosen as

$$\begin{aligned} \sigma_B(v_1 \to v; v_2) &= \frac{1}{v_1 - v_2} \chi_{[v_2,v_1]}(v) \\ \sigma_A(v_1 \to v; v_2) &= \frac{1}{v_2 - v_1} \chi_{[v_1,v_2]}(v) \end{aligned} \tag{2.6}$$

This choice appears more reasonable than the previous one; a driver will now brake to a speed equidistributed between his present speed and the speed of his lead vehicle. Unfortunately, we will see that this model, as many others, possesses only trivial equilibria (this is the reason why in [7] the diffusion term was added).

Even simpler than (2.6) is the choice

$$\sigma_B(v_1 \to v; v_2) = \delta_{v_2}(v), \quad \sigma_A(v_1 \to v; v_2) = \delta_{v_2}(v).$$

In such a perfect world, each driver would be able to assess the speed of his/her leading vehicle exactly and change his/her own velocity with complete precision. For such behaviour, the identity $C(f) = 0$ reduces (after some calculation) to

$$\rho \left[(k-1) \int_v^1 |v_1 - v| F(v_1) F(v) \, dv_1 - (k-1) \int_0^v |v_1 - v| F(v_1) F(v) \, dv_1 \right] = 0,$$

where $k = \frac{p_B q_B}{q_A}$. We see that the left hand side vanishes identically if $k = 1$, so in this case (which is not realistic if there are two or more lanes) every F is an equilibrium. If $k \neq 1$ we have to have

$$\int_0^1 (v_1 - v)F(v)F(v_1)dv_1 = 0,$$

or $F(v)[\int v_1 F(v_1)\, dv_1 - v] = 0$, i.e., $F(v) = \delta_{v_0}(v)$, where v_0 is the average velocity associated to F. Hence all equilibria are trivial.

The previous two choices are special cases of the class of models

$$\begin{aligned} \sigma_B(v_1 \to v; v_2) &= \tilde{f}_{v_1,v_2}(v)\chi_{[v_2,v_1]}(v) \\ \sigma_A(v_1 \to v; v_2) &= \tilde{g}_{v_1,v_2}(v)\chi_{[v_1,v_2]}(v) \end{aligned} \tag{2.7}$$

with families of probability densities $\tilde{f}_{v_1,v_2}$ and $\tilde{g}_{v_1,v_2}$ for the choice of post-braking and post-acceleration speeds. Models of this type look promising inasmuch as there is a lot of generality with respect to average driving behaviour. However, they do contain the assumption that drivers will never underestimate (overestimate) the speed of a leading vehicle in the case of braking (accelerating). For braking, one will always have $v \geq v_2$ (never $v < v_2$, which was possible in the model with σ_A and σ_B given as in (2.4),(2.5)). This is an unrealistic feature of (2.7).

3. Equilibria search for the models (2.7). We are interested in density functions $f(v)$ (measures $\mu(dv)$) such that in the sense of distributions $C(f) = 0$. In view of the definition of $C^+(f)$, this means that

$$k(G_B - L_B)f + (G_A - L_A)f = 0, \tag{3.1}$$

where we have again used the abbreviation $k = \frac{p_B q_B}{q_A}$ (k is a constant for constant ρ). Written out in detail, (3.1) is

$$\rho k \int_{v_2} \int_{v_1 > v_2} |v_1 - v_2| \tilde{f}_{v_1,v_2}(v) \chi_{[v_2,v_1]}(v) F(v_1) F(v_2)\, dv_1 dv_2$$
$$- \rho k \int_{v_2 < v} |v - v_2| F(v) F(v_2)\, dv_2$$
$$+ \rho \int_{v_2} \int_{v_1 < v_2} |v_1 - v_2| \tilde{g}_{v_1,v_2}(v) \chi_{[v_1,v_2]}(v) F(v_1) F(v_2)\, dv_1 dv_2$$
$$- \rho \int_{v_2 > v} |v - v_2| F(v) F(v_2)\, dv_2 = 0.$$

We factor out ρ, multiply with a test function $\varphi \in C([0,1];\mathbb{R})$ and integrate over the admissible speed range $v \in [0,1]$. After some changes in notation (in some of the integrals v is changed to w, and v_1 to v) we obtain

$$\int_0^1 \int_0^v F(v)F(v_2)(v - v_2) \Big[k\varphi(v) + \varphi(v_2) - k \int_{v_2}^v \varphi(w) \tilde{f}_{v,v_2}(w)\, dw - \int_{v_2}^v \varphi(w) \tilde{g}_{v_2,v}(w)\, dw \Big]\, dv_2 dv = 0 \tag{3.2}$$

From (3.2) it is transparent that $C(f) = 0$ admits only trivial solutions if the bracket in (3.2) maintains the same sign for any choice of $0 \leq v_2 < v \leq 1$ and an admissible class of test functions. Let

$$\Phi(v_2, v) := k\varphi(v) + \varphi(v_2) - k \int_{v_2}^v \varphi(w) \tilde{f}_{v,v_2}(w)\, dw - \int_{v_2}^v \varphi(w) \tilde{g}_{v_2,v}(w)\, dw.$$

Clearly, $\Phi(v_2, v)$ depends parametrically on $\tilde{f}_{v,v_2}$ and $\tilde{g}_{v_2,v}$, as probability density models for accelerating and braking, and on k, which depends on the traffic density (recall that large k means high braking probability, i.e., high density, and small k corresponds to a fairly empty highway). We will focus on the case where there is a certain symmetry between braking and acceleration, and there is a self-symmetry of the braking (acceleration) density which gives all the $\tilde{f}_{v_1,v_2}$s and $\tilde{g}_{v_1,v_2}$s in terms of one "generating" density. In this case we will show that all equilibria are trivial.

Our assumptions mean that

$$\text{A.1.} \quad \tilde{f}_{v_1,v_2}(w) = \tilde{g}_{v_2,v_1}(v_1 + v_2 - w),$$

and there is a probability density $\tilde{f} = \tilde{f}_{1,0}$ such that

$$\text{A.2.} \quad \tilde{f}_{v_1,v_2}(w) = \frac{1}{v_1 - v_2}\tilde{f}(\frac{w - v_2}{v_1 - v_2}),$$

where it is understood that $v_1 > v_2$. Figure 1 illustrates the geometric idea behind A.1 and A.2.

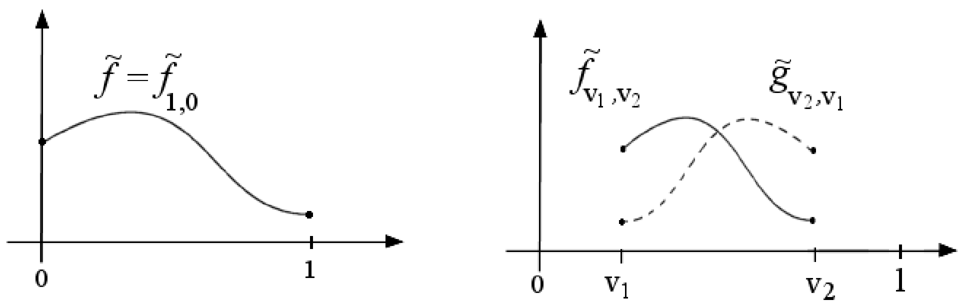

Figure 1. Assumed Scaling and Symmetries

Note that if we abbreviate $\tilde{g} = g_{\tilde{0},1}$, then $\tilde{f}(u) = \tilde{g}(1 - u)$, and by A.1 and A.2

$$\tilde{g}_{v_2,v_1}(w) = \frac{1}{v_1 - v_2}\tilde{f}\left(\frac{v_1 - w}{v_1 - v_2}\right).$$

With these identities, the bracket inside (3.2) becomes

$$k\varphi(v) + \varphi(v_2) - \frac{k}{v - v_2}\int_{v_2}^{v} \varphi(w)\left[\tilde{f}(\frac{w - v_2}{v - v_2}) + \frac{1}{k}\tilde{f}(\frac{v - w}{v - v_2})\right] dw. \tag{3.3}$$

To simplify things further, we rewrite the last integral in the form

$$\int_0^1 \varphi(v_2 + u(v - v_2))\tilde{f}(u)\, du + \frac{1}{k}\int_0^1 \varphi(v - u(v - v_2))\tilde{f}(u)\, du.$$

Dividing by $1+k$ the bracket in (3.2) becomes

$$\begin{aligned}&\frac{1}{1+k}(k\varphi(v)+\varphi(v_2))-\frac{k}{1+k}\int_0^1\varphi(v_2+u(v-v_2))\tilde{f}(u)\,du\\&-\frac{1}{1+k}\int_0^1\varphi(v-u(v-v_2))\tilde{f}(u)\,du.\end{aligned}\tag{3.4}$$

This is a difference of two averages of φ, namely, denoting

$$\mu^1_{v,v_2}=\frac{1}{1+k}(k\delta_v+\delta_{v_2})$$

and

$$\langle\mu^2_{v,v_2},\varphi\rangle=\frac{k}{1+k}\int_0^1\varphi(v_2+u(v-v_2))\tilde{f}(u)\,du+\frac{1}{1+k}\int_0^1\varphi(v-u(v-v_2))\tilde{f}(u)\,du,$$

the difference of averages

$$\langle\mu^1_{v,v_2}-\mu^2_{v,v_2},\varphi\rangle.$$

The measures μ^1_{v,v_2} and μ^2_{v,v_2} are generated by rescaling from $\mu^1:=\mu^1_{1,0}$ and μ^2, given in terms of $\tilde{f}$ by

$$\langle\mu^2,\varphi\rangle=\frac{k}{1+k}\int_0^1\varphi(u)\tilde{f}(u)\,du+\frac{1}{1+k}\int_0^1\varphi(1-u)\tilde{f}(u)\,du.$$

We formulate our first rigorous result on the nature of equilibria in terms of these measures.

Theorem 3.1. *Suppose that $k\neq 1$ and that μ^1_{v,v_2} and μ^2_{v,v_2} satisfy A.1 and A.2. If $\int_0^1 u\tilde{f}(u)\,du<1$, then all kinetic equilibria are trivial.*

Proof. We set $\gamma=\int_0^1 u\tilde{f}(u)\,du$ and choose $\varphi(v)=\alpha+\beta v$, where α and β are constants. We compute

$$\begin{aligned}&(1+k)\langle\mu^1_{v,v_2}-\mu^2_{v,v_2},\varphi\rangle\\=&\beta(kv+v_2)-\beta(k\int_0^1(v_2+u(v-v_2))\tilde{f}(u)\,du+\int_0^1(v-u(v-v_2))\tilde{f}(u)\,du)\\=&\beta(kv+v_2)-\beta(k(v_2+\gamma(v-v_2))+v-\gamma(v-v_2))\\=&\beta[(k-1)v+(1-k)v_2-\gamma(k-1)(v-v_2)]\\=&\beta(k-1)(1-\gamma)(v-v_2).\end{aligned}$$

Inserting this in (3.2) we find

$$\beta(k-1)(1-\gamma)\int_0^1\int_0^v F(v)F(v_2)(v-v_2)^2\,dv_2dv = 0,$$

and the assertion follows.

Remark. It was sufficient to consider linear test functions to prove this result. We expect the result to also hold for $k=1$ and $\gamma<1$, but one has to use at least quadratic polynomials for a proof. As the importance of this case is small, we skip these calculations. The case $\gamma=1$ is the Delta-function case.

We conjecture that Theorem 3.1 generalizes to all cases where drivers never underestimate (overestimate) the speed of a slower (faster) lead vehicle, with the possibility of some singular exceptions. As these assumptions on driver behaviour are unrealistic we make no further attempts to generalize the above result.

Remark. A question of largely academic interest in this context is the stability of these trivial equilibria.

4. Inclusion of relaxation terms. The problem of nonexistence of smooth equilibria can be alleviated by augmenting the collision operator with relaxation terms. Addition of such terms to the collision operator may seem somewhat arbitrary and unjustified; however, in the next section we will show that realistic assumptions on the probability densities σ_A and σ_B lead automatically to such relaxation terms.

Relaxation terms will not only enforce the existence of nontrivial equilibria $f^e = f^e(\rho; v)$; in simple situations, f^e can be computed explicitly. This was done in [7] for the special relaxation term from (2.2). Here, we review the method and generalize it to relaxation terms

$$\nu(\rho)(G_s - L_s)(f)$$

with $G_s(f) = \int_0^1 f_0(v) f(v_2)\, dv_2, \quad L_s(f)(v) = f(v)$. The rationale is that in the class of probability densities we will have $G_s(f) = L_s(f)$ if and only if $f(v) = f_0(v)$, so f_0 can be interpreted as the natural velocity distribution density according to which drivers will choose their speed if free driving is possible.

The explicit calculation of $f^e(\rho, v)$ by an ODE method is possible for

$$\sigma_A(v_1 \to v; v_2) = \frac{1}{v_2 - v_1} \chi_{[v_1, v_2]}(v)$$
$$\sigma_B(v_1 \to v; v_2) = \frac{1}{v_1 - v_2} \chi_{[v_2, v_1]}(v).$$

In this case

$$C(f) = q_A[k(G_B - L_B) + (G_A - L_A) + \frac{\nu}{q_A}(G_s - L_S)], \tag{4.1}$$

where, as before, $k = P_B q_B / q_a$ and in view of (2.1)

$$\begin{aligned} G_B = G_A &= \rho \int_0^1 \int_{v_2}^1 F(v_1)F(v_2)\chi_{[v_2,v_1]}(v)\, dv_1 dv_2 \\ &= \rho \int_0^1 \int_0^{v_1} F(v_1)F(v_2)\chi_{[v_2,v_1]}(v)\, dv_2 dv_1 \\ L_B &= \rho \int_0^v F(v)F(v_2)(v - v_2)\, dv_2 \\ L_A &= \rho \int_v^1 F(v)F(v_2)(v_2 - v)\, dv_2 \\ G_s &= \int_0^1 f_0(v)F(v_2)\, dv_2 \\ L_s = F(v) &= \int_0^1 F(v)F(v_2)\, dv_2. \end{aligned}$$

Following [7] we renormalize further by setting $\kappa = \frac{k}{1+k}$. Then $C(f) = 0$ is equivalent to

$$\kappa(G_B - L_B) + (1 - \kappa)(G_A - L_A) + \frac{\nu}{q_A(1+k)}(G_s - L_s) = 0.$$

We can factor out ρ.

Let us introduce the new dependent variable $p := \mathbf{F}(v) = \int_0^v F(w)\,dw$. $\mathbf{F} : [0,1] \longrightarrow [0,1]$ is monotone increasing. If it is 1-1, then $v = \mathbf{F}^{-1}(p)$, and

$$1 = \frac{dp}{dp} = \frac{d\mathbf{F}(v(p))}{dp} = F(v(p))\frac{dv}{dp},$$

so $\frac{dv}{dp} = \frac{1}{F(v(p))}$. We abbreviate $\dot{v} = \frac{dv}{dp}$.

The various parts of the collision term now become

$$\begin{aligned} G_B &= G_A = p(1-p) \\ L_B &= F(v)[v \int_0^v F(w)\,dw - \int_0^v wF(w)\,dw] \\ &= \frac{1}{\dot{v}}[v(p)p - \int_0^p v(q)\,dq] \end{aligned}$$

(here, we have substituted $w = w(p(w)), \quad dp(w) = F(w)dw$)

and similarly

$$\begin{aligned} L_A &= \frac{1}{\dot{v}}\left(\int_p^1 v(q)\,dq - v(p)(1-p)\right) \\ G_s &= \tilde{f}_0(p), \text{where } \tilde{f}_0(p) := f_0(v) = f_0(v(p)), \\ L_s &= \frac{1}{\dot{v}(p)}. \end{aligned}$$

The identity $C(f) = 0$ becomes

$$\begin{aligned} &\dot{v}p(1-p) - \kappa\left[pv(p) - \int_0^p v(q)\,dq\right] \\ -(1-\kappa)&\left[\int_p^1 v(q)\,dq - v(p)(1-p)\right] + c\dot{v}\left[\tilde{f}_0(p) - \frac{1}{\dot{v}}\right] = 0, \end{aligned}$$

with $c = \frac{\nu}{\rho q_A(1+k)}$. Differentiation with respect to p reduces this to the second order ODE

$$\frac{d}{dp}\ln\dot{v}(p) = \frac{3p + \kappa - 2 - c\frac{d}{dp}\tilde{f}_0(p)}{p(1-p) + c\tilde{f}_0(p)}$$

with boundary conditions $v(0) = 0, \quad v(1) = 1$. For $f_0(v) = 1$, this is explicitly solved in [7] and leads to a parametric representation of equilibria. In this case, the ODE is explicitly integrable by using partial fractions. This method certainly applies for simple probability densities $\tilde{f}_0(p)$.

For other choices of f_0 the boundary value problem has to be numerically integrated. We investigate $f_0(v) = 2v$, $f_0(v) = 2 - 2v$ and $f_0 = \frac{\pi}{2} sin(v\pi)$.

Figure 2 shows plots of the equilibrium solutions for $\kappa = 0.5$.

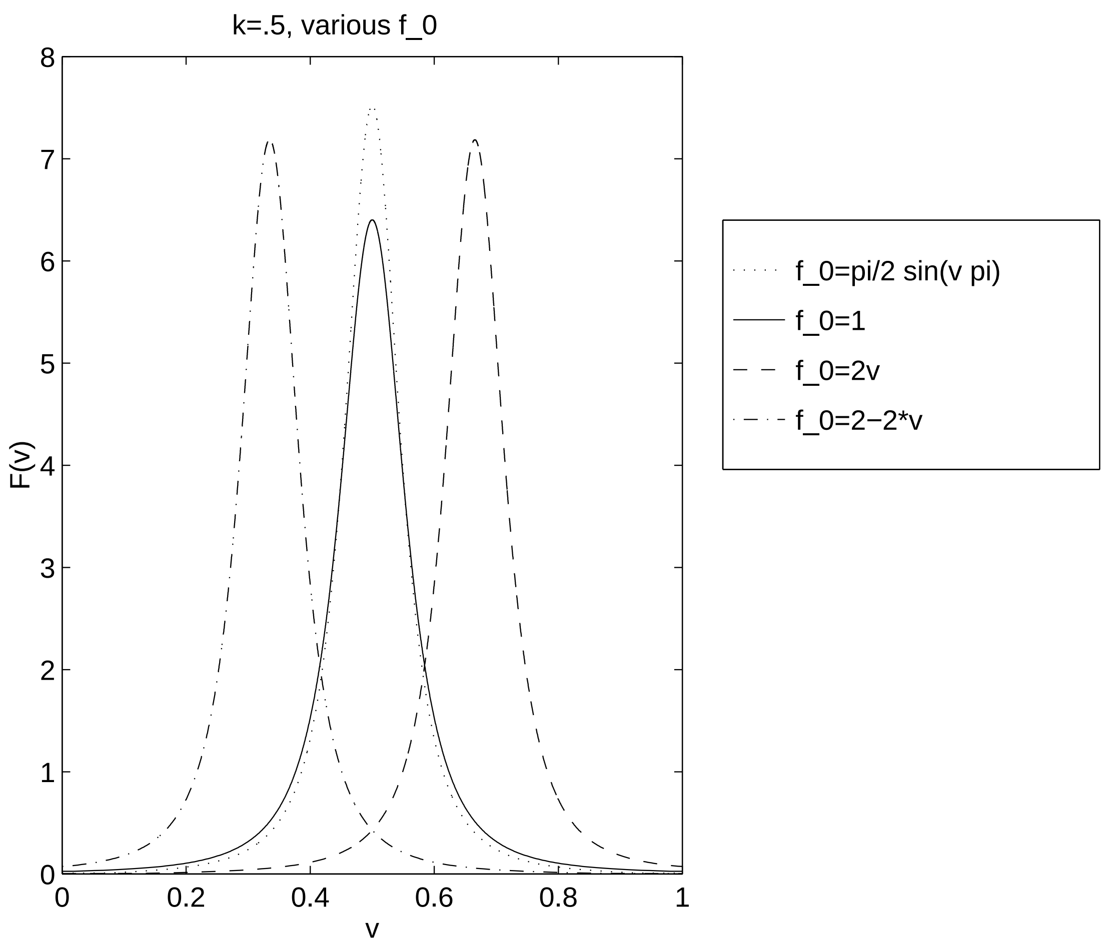

Figure 2.

5. A possible origin of diffusion. We saw in Section 3 that under reasonable assumptions on the $\sigma(v_1 \to v; v_2)$ probability densities only trivial equilibria will exist if drivers will never brake below (accelerate above) the speed of their leading vehicle. If Section 4 we added artificial diffusion to the collision operator and saw that this not only enforces the existence of nontrivial equilibria; for the simple σs given by (4.1), we were actually able to compute these equilibria explicitly by ODE methods.

We suggest that realistic σs will contain a component which will generate a diffusion term as part of the collision operator (similar to the one which was added artificially in Section 4) automatically. As shown by numerical experiments in [2] and analytical arguments in [8], this statement applies to the σ_B and σ_A as given by (2.5). However, these σs are independent of the speed of the leading vehicle and therefore unrealistic. We now offer an alternative.

To start, consider the situation where $v_1 = 1$ and $v_2 = 0$, i.e., the lead vehicle is stalled and the driver under consideration is moving at the speed limit. We suggest that in this situation σ_B is given by

$$\sigma_B(1 \to v; 0) = \alpha\delta_0(v) + (1-\alpha)\tilde{f}(v), \tag{5.1}$$

where α is the probability of a full stop, and $\tilde{f}$ is the probability density of the residual speed if no full stop is made. We will call $\tilde{f}$ the "residual braking density." A sketch of this σ_B is given in Figure 3.

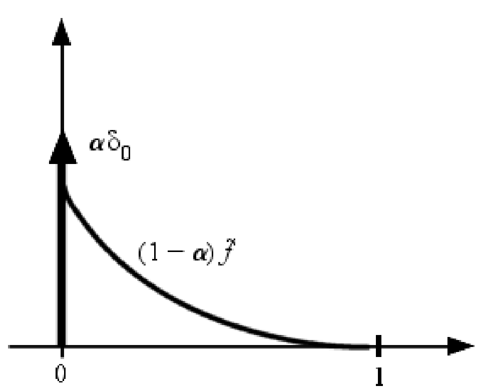

Figure 3. σ_B from (5.1)

In section 3 we considered the case where the probability density $\sigma_B(v_1 \to v; v_2)$ is obtained from $\sigma_B(1 \to v; 0)$ by rescaling relative to the interval $[v_2, v_1]$. We saw that this leads only to trivial equilibria.

A closer look implies that such simple rescaling of $\sigma_B(1 \to v; 0)$ is inappropriate, because it would entail that fraction α of all drivers with speed v_1 would break *exactly* to v_2. This is unrealistic—humans don't have the ability to estimate the speed of a leading vehicle with such accuracy (except when v_2=0). To accomodate for this weakness, we assume that there is a smooth probabililty density ω_B with $\operatorname{supp} \omega_B \subset [-1, 1]$ such that for $v_2 > 0$

$$\sigma_B(v_1 \to v; v_2) = \frac{\alpha}{v_2}\omega_B\left(\frac{v - v_2}{v_2}\right) + (1-\alpha)\frac{1}{v_1 - v_2}\tilde{f}\left(\frac{v - v_2}{v_1 - v_2}\right)\chi_{[v_2, v_1]}(v). \tag{5.2}$$

See Figure 4.

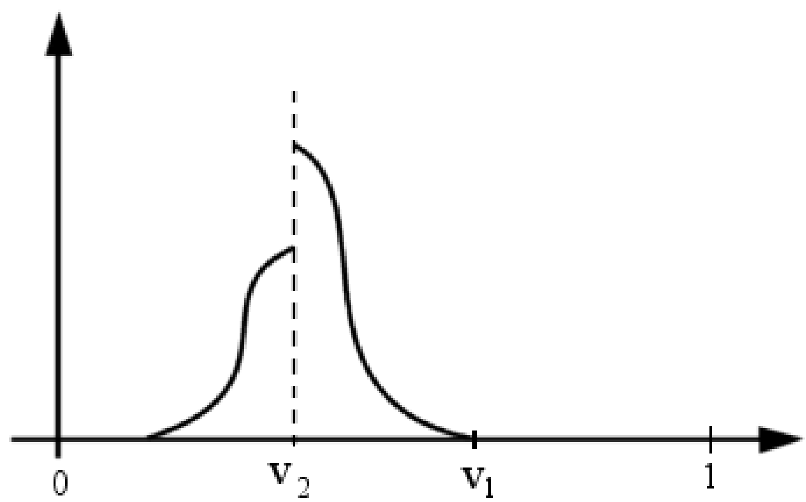

Figure 4. σ_B from (5.2)

Note that as $v_2 \searrow 0$ $\frac{\alpha}{v_2}\omega_B\left(\frac{v-v_2}{v_2}\right) \longrightarrow \alpha\delta_0(v)$ in the sense of distributions. This part of σ_B scales relative to v_2, while the residual part $\frac{1}{v_1-v_2}\tilde{f}\left(\frac{v-v_2}{v_1-v_2}\right)$ scales relative to $v_1 - v_2$, as before.

Strictly speaking we have to modify (5.2) somewhat further: for $v_2 > 1/2$

the term $\frac{\alpha}{v_2}\omega_B\left(\frac{v-v_2}{v_2}\right)$, as it stands, predicts contributions beyond the speed limit. This is a trivial difficulty which is easily addressed by an additional trunction.

We suggest a similar mechanism for the acceleration contributions, i.e., that for some $0 < \beta \leq 1$

$$\sigma_A(0 \to v; 1) = \beta\delta_1(v) + (1-\beta)\tilde{g}(v) \tag{5.3}$$

and for $v_1 < v_2$

$$\sigma_A(v_1 \to v; v_2) = \frac{\beta}{1-v_2}\omega_A\left(\frac{v-v_2}{1-v_2}\right) + (1-\beta)\frac{1}{v_2-v_1}\tilde{g}\left(\frac{v-v_1}{v_2-v_1}\right)\chi_{[v_1,v_2]}(v). \tag{5.4}$$

The residual acceleration density $\tilde{g}$ scales as in Section 3, and as the residual braking density introduced earlier, while the singular part of σ_A scales relative to $1 - v_2$.

Example. A simple choice, which will be used for the numerical computations, is

$$\tilde{f}(v) = 1, \quad \omega_B(v) = \chi_{[-1,0]}$$

and

$$\tilde{g}(v) = 1, \quad \omega_A(v) = \chi_{[0,1]}$$

Remark. It is debatable whether the scaling of the singular part in (5.4) should be as defined. (5.3) predicts that a fraction β of all drivers have the ability to identify the speed of a leading vehicle moving at the speed limit with accuracy. However, this is far less realistic than the braking scenario where one encounters a stalled vehicle. For the present paper we will continue to use (5.4) and write down the emerging diffusion terms. We assert that similar terms will arise in any modeling effort which contains a realistic ansatz for σ_A.

Consider now the braking terms of the collision operator. In the notation of Section 3

$$
\begin{aligned}
&k(G_B - L_B)\\
=&k\rho\Big(\int_0^1 \int_{v_2}^1 |v_1 - v_2|\sigma_B(v_1 \to v; v_2)F(v_2)F(v_1)\, dv_1 dv_2\\
&\qquad - \int_0^v |v - v_2|F(v)F(v_2)\, dv_2\Big)\\
=&k\rho\left[(1-\alpha)\int_0^1 \chi_{[v_2,v_1]}(v)\tilde{f}\left(\frac{v-v_2}{v_1-v_2}\right)F(v_2)F(v_1)\, dv_1 dv_2\right.\\
&\qquad \left. - (1-\alpha)\int_0^v |v - v_2|F(v)F(v_2)\, dv_2\right]\\
&+ k\rho\left[\alpha\int_0^1\int_{v_2}^1 |v_1 - v_2|\frac{1}{v_2}\omega\left(\frac{v-v_2}{v_2}\right)F(v_2)F(v_1)\, dv_1 dv_2\right.\\
&\qquad \left. - \alpha\int_0^v |v - v_2|F(v)F(v_2)\, dv_2\right].
\end{aligned}
\tag{5.5}
$$

The first two terms in the right-hand side correspond to gain and loss due to braking, with a braking kernel σ_B of the type discussed in Section 3 (where we saw that such models only admit trivial equilibria). The last two terms conspire to produce a diffusion term. We rewrite this term (omitting the factor $k\rho$)

$$
\begin{aligned}
&\alpha\int_0^1 \left(\int_0^{v_1} |v_1 - v_2|\frac{1}{v_2}\omega\left(\frac{v-v_2}{v_2}\right)F(v_2)\, dv_2\right)F(v_1)\, dv_1\\
&\qquad - \alpha\int_0^v |v - v_2|F(v)F(v_2)\, dv_2.
\end{aligned}
$$

This is more complicated than the simple diffusion terms from Section 4. In particular, it is evident that trivial equilibria still exist (this is reasonable, as they exist in reality), but they are not stable: for smooth perturbations of $\delta_{v_0}(v)$, i.e., $F = \delta_{v_0}(v) + \epsilon\varphi(v)$, the effect of φ will be amplified by the diffusion term. The system will be forced towards a smooth nontrivial equilibrium (whose existence is plausible but very hard to prove analytically).

A similar discussion applies to the acceleration terms. With the ansatz (5.4),

$$
\begin{aligned}
&G_A - L_A\\
=&\rho\left[(1-\beta)\int_0^1\int_0^{v_2}\chi_{[v_1,v_2]}(v)\tilde{g}\left(\frac{v-v_1}{v_2-v_1}\right)F(v_2)F(v_1)\,dv_1dv_2\right.\\
&\qquad\left.-\ (1-\beta)\int_v^1|v-v_2|F(v)F(v_2)\,dv_2\right]\\
+&\rho\left[\beta\int_0^1\left(\int_{v_1}^1\frac{v_2-v_1}{1-v_2}\omega\left(\frac{v-v_2}{1-v_2}\right)F(v_2)\,dv_2\right)F(v_1)\,dv_1\right.\\
&\qquad\left.-\int_v^1|v-v_2|F(v)F(v_2)\,dv_2\right].
\end{aligned}
\tag{5.6}
$$

The last two terms in (5.5) and the last two terms in (5.6) together comprise a diffusion component of the collision operator which suffices to steer the flow pattern away from trivial equilibria.

The observations of this section suggest a more systematic search for the *real* σs. It would be an interesting and potentially rewarding project to investigate to what extent the shape and stability of emerging equilibria depends on the ansatz for the σs. Such an investigation can probably only be done numerically, but it would have to be done only once to arrive at equilibria which could then be used for macroscopic models, as discussed in [2] and [7].

We conclude with a numerical investigation. The time dependent spatially homogeneous problem was solved numerically to determine the stationary distributions. This was done by a finite volume method which ensures particle conservation.

We consider σ_B and σ_A as given in (5.2) and (5.4). The artificial diffusion term is set to be 0.

The functions $\tilde{f}, \tilde{g}$ and ω_B, ω_A are chosen as in the example: $\tilde{f}(v) = 1$, $\omega_B(v) = \chi_{[-1,0]}$ and $\tilde{g}(v) = 1$, $\omega_A(v) = \chi_{[0,1]}$. Different values of $\kappa := k/(1+k)$ and α, β are investigated:
Figure 5 shows $\kappa = 0.5$ (i.e., $k = 1$), $\alpha = \beta = 0.05$ and $\alpha = \beta = 0.2$.

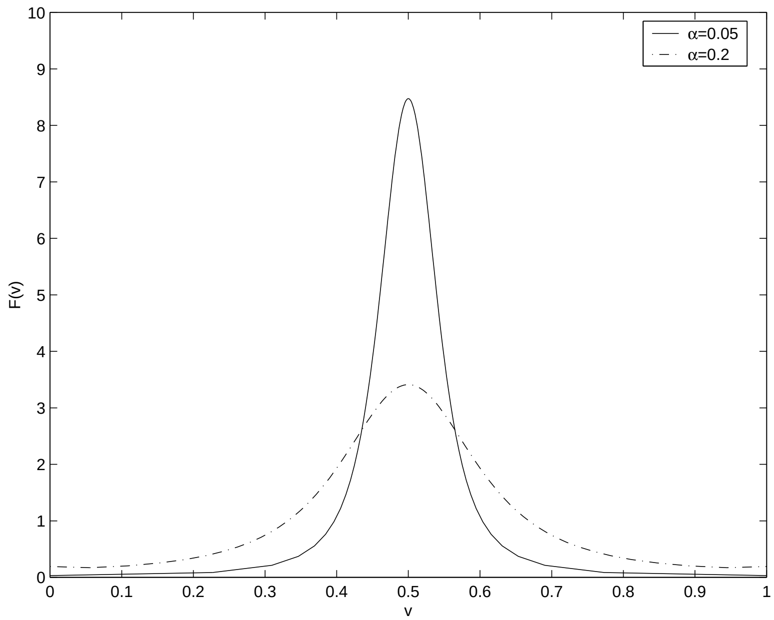

Figure 5: $\kappa = 0.5$

Figure 6 shows $\kappa = 0.35$ (corresponding to a smaller braking probability), $\alpha = \beta = 0.05$ and $\alpha = \beta = 0.2$.

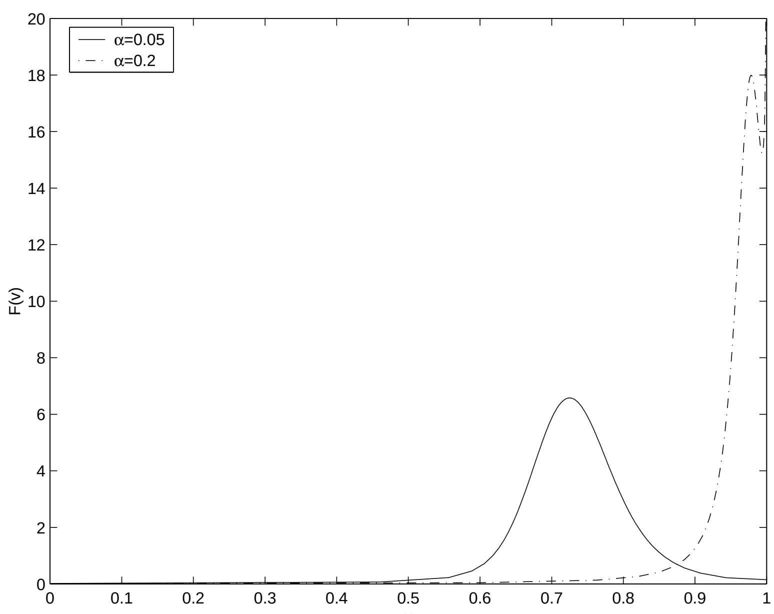

Figure 6: $\kappa = 0.35$

For $\kappa = 0.65$ we would obtain a figure which is symmetric to Figure 6 with distribution functions concentrated near 0 instead near 1.

We note that for $\alpha = 0.2$, i.e. stronger singular braking and acceleration behaviour, we obtain bimodal equilibrium distributions.

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