Strategy, Conflict and Cooperation

David Scoones
Department of Economics
University of Victoria
Chapter 1: Foundations

“When I use a word it means just what I choose it to mean, nothing more nor less”

*Humpty Dumpty*

This lecture introduces the basic elements of game theory used for the rest of the course. The focus here is on constructing *well-formed* games; the next lecture introduces some ways to *solve* games. This lecture contains the basic vocabulary that you need to make sense of the formal games you will meet in this course. Even though you will meet unfamiliar terms, you should pay careful attention to the meaning of common words in Standard English. Typically, the “formal” meaning in game theory is more restrictive, taking only one of the possible meanings of the word in everyday English. This restrictiveness is essential for our objective of making clear, unambiguous statements about strategic situations. Misinterpret words and our careful construction of a game will amount to nothing. Like Humpty Dumpty we will be particular in our use of words, but perhaps less subjective, deferring in our choice to the standard lexicon of game theorists.

Game theory is a language for describing and analyzing strategic situations. Its practical value rests entirely on the accuracy of the description. Description involves and requires making “assumptions”. Unfortunately, assumptions can rarely be read from reality, based on directly confirmable facts about the world, upon which all observers agree. (Such cases do exist.) Largely, what distinguishes a good game theorist from a poor game theorist is how well these basic modeling choices are made. Well-chosen assumptions lead game that is tractable (i.e. feasible to solve) and captures the important features of the situation under study. Good game theorists are also skillful solvers of games. This too is sometimes challenging. But many extremely useful games are not at all hard to solve, once specified correctly. Of course, the worst situation is a poorly chosen set of assumptions, which are then analyzed incorrectly. This is more common than it should be. Even when, through a stroke of luck, bad reasoning arrives at the correct conclusion, the invalid and irrelevant model is at best technical window-dressing, a mind-clouding distraction.

You must understand that “poorly chosen” isn’t the same thing as wrong. The best models contain assumptions that do not describe reality exactly, and are thus in some sense “wrong.” If this were not so, the model would be as intractable as the reality, so it wouldn’t help much. Nevertheless, there is no virtue in error for its own sake, and a close approximation to reality is generally desirable. The emphasis is on finding the “important features” of the situation. You would find it easier to follow a map to my
house that featured roads labeled with names, rather than shrubbery, no matter how realistically drawn. “Turn left at the trees” is generally poor guidance.

Throughout the course you should maintain an open mind about assumptions. Ask yourself: What assumptions are crucial to the result presented? How appropriate are these assumptions for the situation under consideration? What would change if these assumptions were altered? The language and concepts in this lecture have been created to facilitate exactly this process.

For most of the course behavior is assumed to be “rational.” This lecture explains and illustrates what this term means\(^1\). Rationality isn’t required by game theory, and other approaches to strategic choice exist. Later in the course we will explore one alternative, Evolutionary Game Theory, which reinterprets and reuses many of the concepts developed in this lecture. You will see that these approaches are complementary.

Key terms: Choice, ordinal payoff, strategy, extensive form, normal (strategic) form. Additional key terms are noted by **bold font**.

### 1. Rational Choice Theory

A model of choice links a set of choices or actions to a set of outcomes (or consequences). The link may be deterministic, where each choice leads with certainty to a known outcome, or stochastic, where chance partially determines the outcome. Simple (one person) Rational Choice Theory sets aside strategic considerations by assuming that no other decision-makers are involved. (Sometimes, however, we speak as if chance itself were an actor, usually called nature. In such cases, we assume that nature is not motivated by any interest in the outcome.)

Example: Do you have another cup of coffee? Actions: another cup, not another cup. Consequences: if you have another cup, you have some pleasure, perhaps, and face financial, time and possible health costs; if you don’t you forego pleasure, possibly, but avoid those negative costs. This may be entirely deterministic, or you may believe that 15 minutes more or less drinking coffee at the lunch counter in Schwab’s Pharmacy can change your life forever.

Example: The Monte Hall Problem. A game show contestant is presented with three doors. Behind one is a valuable prize, behind the other two, nothing. The contestant makes a preliminary choice and then the host (Monte Hall) opens one of the other doors. Monte will only open a door that does not conceal the prize: he can always do this since at least one (and perhaps both) of the doors not chosen by the contestant will not hide the prize. The contestant is then offered the choice of either remaining with her first pick or switching to the other closed door. Should the contestant switch?

---

\(^1\) Remember: here and, unless explicitly stated, elsewhere, when I say “means” I refer to its use in game theory, not the “true” meaning of the word, whatever that means. Okay, that last one was an exception.
Payoffs
Choice theory assumes that decision-makers make the choice that leads (or is likely to lead) to their most preferred outcome. Choice theory represents preferences by assigning numbers to the outcomes. Higher numbers indicate “more preferred” outcomes. In choice theory, the quantity measured by the numbers assigned to outcomes is usually called utility. In game theory, it is usually called a payoff. In some cases, this assignment is very natural, for example, when the outcomes are amounts of money or years of life. In others it is less so, and sometimes you might even think that a numerical assignment to represent preferences is impossible or at least immoral. Despite the interesting and significant psychological and philosophical issues involved, we will mostly just assume without comment that an assignment can be made.

Ordinal and Cardinal Utilities
If nothing more than a ranking is implied by the numbers assigned to outcomes to represent preferences, the utilities are said to be ordinal. For example, if having another cup of coffee is more satisfactory than having a cup of tea, utilities could be represented by any two numbers where the higher number associated with having coffee and the lower with having tea. Nothing but the relative size of the numbers is relevant. In particular, an ordinal ranking says nothing about how much better you feel having the coffee. Ordinal rankings use words like “higher”, “faster” and “stronger”. (Olympic medals are awarded on an ordinal ranking.)

We may also be interested in how much better one outcome is than another. This is so-called cardinal information. Profits and commuting time are cardinal: you can, in a very real sense, earn twice the money from one choice as from another, or get home in one third the time on one route as on another. At first glance, it is not so clear what it means to say that drinking coffee is twice as satisfying as drinking tea. It turns out that cardinal information is contained in preferences more frequently than you might expect. When outcomes are subject to random factors, depending, for example, on the role of dice or the toss of a coin, preferences can be represented as cardinal. Even though saying a cup of coffee is twice as satisfying as a cup of tea seems strange, it is plainly true that a person might be indifferent between cup of tea and 50% chance of getting
coffee (and 50% of getting nothing) based on a coin toss. For this to be represented, the utility numbers must themselves contain cardinal information. We will return to this subject, briefly, later in the course when we introduce randomization.

Whether cardinal or ordinal information is required, there always exist multiple ways of representing preferences. How much flexibility you have depends on whether the representation is cardinal or ordinal, but the same preferences can always be represented by more than one assignment of numbers. We generally use the simplest representation. If only the relative size matters, we can use any numbers that maintain the ordinal ranking of outcomes. Seen for the first time, the assignment of utilities can seem quite arbitrary and disconcerting. Don’t be put off: while it is never true that “anything goes,” the choice of representations is to a certain degree arbitrary.

**Assigning Preferences**

As mentioned, selecting a particular set of numbers to represent utility is simple in some cases. For example, for a business seeking to earn the highest possible profits, utility can be identified directly with profit. Or, the utility for a commuter trying to get home as quickly as possible can be represented as the negative of time spent travelling. (Negative because less time is better, and negative numbers are less than positive ones.) In these cases, you might say a “natural assignment” exists. All that really matters is that the utilities represent the preferences of the decision makers in your model. Choice theory is about how decision makers should make decisions, not about what they should prefer. We will in general make few judgments about what decision makers should want. Instead, we aim to represent what they do want.

---

**What’s rational about Rational Choice Theory?** Revealed preference theory models preferences as “revealed” by choices. It neither requires nor denies an underlying psychological structure. The theory assumes that choices are logically and consistently directed toward achieving the decision maker’s goals, whatever those happen to be. Revealed preference theory imposes “rationality” by required that decisions be consistent across sets of alternatives. Among other things, the theory rules out the possibility of cycles in preferences, where, for example, someone prefers “driving” to “riding the bus”, and “riding the bus” to “walking,” but prefers “walking” to “driving”. Assigning utility numbers, ordinal or cardinal, imposes this consistency with the mathematical operation of inequalities.

This consistency requirement is a narrow use of the word rational. Some people believe that “Truly Rational” behavior must have some additional substance. It must be “good” for the decision maker, or even “good for society,” not just congenial, pleasant, or consistent. This is more than we want to put into our theory. Such a substantive notion of rationality assumes away many of difficulties and apparent paradoxes of choice theory. Our goal is to understand how “rational” behavior may lead us astray, so we don’t want to assume too much up front. You may think we should use a different term (e.g. consistent). So long as you keep in mind that we use the term narrowly, you will likely be okay. But be aware that some people may contest our use of the word.
Knowledge

We must also assume the decision maker has enough knowledge to make a decision. For example, they must know what choices are open to them, how these choices lead to various outcomes, and understand their preferences over those outcomes. In reality, decision makers can be wrong on each of these points. For example, “stochastic” choice situations introduce randomness in the form of “nature” making choices and complicating the connection between the decision maker’s actions and the outcomes. In this case, we need to assume the decision maker has some idea of how this randomness works. They could easily misunderstand this, as humans typically do. Similarly, decision makers could be confused about their preferences or the options open to them. Models can reflect these informational problems too, but in doing so they still must carefully specify what the decision maker does know. Assuming total ignorance is usually as uninteresting as assuming complete omniscience is unrealistic.

2. Game theory

Game theory extends simple (individual) rational choice theory to account for the fact that, in general, outcomes depend on the choices made by more than one decision-maker. To use game theory you must first translate the situation of interest into a model (henceforth called a game), and second solve the game. The second step is very close to a science: once a game is well-specified, game theory has explicit rules for how to solve it. You can learn those rules and solve a game in a way that is objectively (or at least inter-subjectively) “correct”. (To be honest, in tricky situations exist in which the solution rules themselves are pretty hotly contested, and different solution methods coexist. These areas constitute the research frontier of game theory. How “scientific” any of this is isn’t something that need occupy our attention.) On the other hand, the first step of translating a situation into a specific game is more of an art. As in all arts some people are truly gifted, but everyone can benefit from practice, and only practice can lead to mastery.

The rest of this lecture focuses on formally defining games. These are the elements needed to write down a game. The next lecture takes up solving them. For now, the one thing you need to know about solutions to games is that, like decision theory, game theory assumes players will seek their most preferred outcome.

A game is formally defined by a set of required features:
1. the set of players,
2. the set of actions players choose among,
3. the set of outcomes that result from the choices of players,
4. the order in which the players move,
5. the information players have about the game, and
6. the players’ preferences over outcomes.
In games, the utility values assigned to outcomes are usually referred to as **payoffs**. The actions and preferences are exactly as described above for choice theory.

Now that we have other players involved, a word about self-interest. Game theorists frequently assume that player preferences are “selfish.” So standard is this assumption, you might mistakenly take it as a required feature of game theory, but it is not. **Something** needs to be assumed: selfishness is often interesting and maybe even often realistic, but it is not necessary. If you decide to build a theory based entirely on altruism, you can (once you settle on one of the various forms of altruism on offer). Of course, doing this means, among other things, that cooperation is assumed rather than explored and explained. More interesting is to understand how selfish motives can lead to apparently altruistic behavior. Moreover, an excess of altruism does not typically pose problems that people wish to understand, let alone “solve.” Where altruism is realistic, game theorists are unlikely to be required.

While the assumption of selfishness is standard, it is also flexibly extended as required. If a mother desires to provide for her children, as many do, we need not assume otherwise. Like pretty much everything else about preferences except inconsistency this can be easily accommodated, provided we assign payoffs correctly. We might ask where such preferences come from, but we do not need to pretend they do not exist. Game theorists modeling families routinely make the assumption that parents value their children. In fact there are good evolutionary reasons why humans care about the payoffs to their children that result from their choices. (We will come to this point again toward the end of the course when we look at evolutionary game theory.)

In addition to evolutionary game theory, there is a growing body of research into “behavioural game theory” that relaxes theoretical restrictions on “rationality” in favour of empirically grounded rules of behavior, typically based in experiments. This literature is closely related to psychology, and much of it is motivated by apparent paradoxes of rational decision theory. While we can often capture the results of experiments with numbers assigned to outcomes, we may have trouble doing so in a simple, easily generalized and consistent fashion. Context matters, for example. And people seem more altruistic toward total strangers than the simplest selfish models suggest they should be. The contrasting difficulty is coming up with a model of behavior general enough to use in the place of rational choice theory. This is a very active area of research.

However, once payoffs are assigned in a game, it is critical that you interpret them correctly. For example, it makes no sense to write down payoffs for a “selfish” parent that are highest when she consumes her child’s dinner, and then say that she wouldn’t prefer this outcome. If a parent feels altruistic to her children, then her payoffs should reflect that, and increase with the welfare of the child. Any similar “interdependence” in preferences must be reflected in the payoffs, not added later as a “*reason*” why people do not choose the highest payoff action\(^2\). Once this is done, we interpret them as always: higher payoffs are attached to more preferred outcomes.

\(^2\) Here I am using the prefix * to indicate incorrect (or at least non-standard) usage
Common Knowledge

For most of the course, we will maintain the assumption that players are rational and know the structure of the game they are playing. To decide what to do, players reason about the game. From the perspective of one player, all players are part of the game. To account for the behavior of other players, the rationality of players is itself something that must be known. Imagine instead that I were not sure you could be counted on to act “rationally”, i.e. in a way consistent with your preferences. Then my knowledge of your preferences wouldn’t serve much purpose, would it? What should I expect you to do, and therefore what should I myself do would be unclear. If I know that you are rational (among other things) I can use this as a guide for predicting your choices.

Interestingly, with interdependent decisions knowledge takes on additional dimensions. It is not (generally) enough for players to know something: they must also know that other players know it. But this first step isn’t enough: when making a choice, I will not only need to know you are rational, for example, I will need to know you know that I know this. Suppose not. Say I knew you knew me to be rational, but was not sure that you knew that I knew this. Then I might wonder if you, thinking incorrectly that I believe I’m irrational, might think that I will choose what is best, given that I think you believe I’m irrational. What to do? In fact, this is still not good enough: to ensure the logic does not founder, we need to assume that these rounds of knowledge go on forever: I know that you know that I know that you know that…. When something is known like this, it is called common knowledge. We will assume that both the rationality of the players and the structure of the game they are playing is common knowledge.

3. Game Forms

Configuring these essential features of a game is what constitutes “writing down” a game. Games can be written down in either an extensive form or a strategic form (the latter is sometimes called a normal form). Each form can be useful depending on the game and the purpose of the representation. It’s not that some situations are extensive form games and other situations are normal form games: this is a modeling choice, based on which form is more useful for the purpose at hand. With some effort, it is always possible to re-write an extensive form game in the strategic form, or vice versa. This point is discussed further below.

The Extensive Form

Extensive form games are generally written as graphs that take the form of a tree. Each decision node on the graph represents a single point of choice for one of the players. Decision nodes are labeled with the name of the player who must choose at that point in the game. Leading from the decision nodes are branches that represent the actions available to the player at the decision node. Branches are labeled with the action taken to move onto the branch. In a given play of the game, choices are made sequentially,
and play moves along the branches of the extensive form, tracing a path through the game tree. Each path represents a possible play of the game. For games with finite number of moves, every path leads to a terminal node containing a list of the payoffs to the players whose collective choices have followed the path.

In the example below, the players are called player 1 and player 2. The decision nodes are represented by circles inscribed with the name of the player whose turn to move at that node. The branches of the tree sprouting from the nodes are labeled with the actions available to the player choosing at the node, here the generic labels “up” and “down”. The final labels at the terminal nodes represent the payoffs. By convention, the first is the payoff for player 1 and the second for player 2. Of course, even though the picture suggests otherwise, there is no reason that the payoffs at every terminal node would be the same (which would make for a very uninteresting game).

**DEFINITION:** A strategy is a list of actions that describe a choice at every point that the player is required to move in any possible play of the game. That is, a strategy specifies a choice at each and every decision node assigned to a player.

In the game above, player 1 has two strategies. Since this player only moves once, her strategies are simply her choices at that first decision node {Up, Down}. Player 2 has four strategies, each of which is contingent on the choice made by player 1. For example, one of player 2’s strategies is: Choose up if player 1 chooses up, and Choose up if player 1 chooses down. This strategy lists what would be done at each (i.e. both) of the decision nodes assigned to player 2. Of course, in any particular play of the game, only one of these nodes will be reached, so “one half” of the strategy is not used. But it must be specified: the list must be complete. The partial “plan” to “choose up, if player 1 chooses up” is not a strategy.

Contingent strategies (such as player 2’s) are a list of “if...then...” statements, where the list of “ifs” must include all decision nodes where the player might ever find herself in any possible play of the game. The “then” is a choice that is available to the player under the circumstances created by the list of “ifs”. A shorthand notation for contingent strategies denotes, for example, the strategy of player 2 mentioned above as (Up|Up) and (Up|Down). These are read “up conditional on up AND up conditional on down”.

©David Scoones 2007-17
(Notice this reverses the order of the “if ...then...” structure; the if condition is after the vertical line.) Clearly, this strategy could be more simply described by the statement Always Up.

Exercise: list all four of player 2’s strategies.

A strategy for a player must indicate a choice at each node assigned to the player, but, as in the example above, not all of these choices will be faced during a particular play of the game. This is because every prior move typically eliminates branches of the game tree. This completeness is absolutely required, but sometimes seems to defy common sense. For example, a player’s choices must be specified even for nodes where a prior move by that same player prevents the node from being reached. This may seem a bit strange. For example, you might think it peculiar that a military strategy calling for a certain country to not be invaded must also specify how, if an invasion were to occur, resistance to the invasion will be overcome. The strategy itself rules out that contingency, so why must it then account for it! We will take up this point in the next lecture when we solve some games, and maybe it will seem a bit less odd.

Information Sets

The game illustrated in the previous section assumes that player 2 knows which action player 1 chose at the first decision node. This “information” allows player 2 to take an action contingent on player 1’s actual choice. In many strategic situations this very precise information about prior moves will not be known. For example, a business setting prices for advertisement in a weekend newspaper might not know the prices set by competing firms. (This is sometime true even when the competitors have already made those choices.) Or a military commander selecting on which of two beaches along a coastline to land her troops may not know how the opposing general has chosen to distribute the defending force.

This lack of information regarding prior play can be represented in the extensive form through the use of information sets. An information set is a collection (set) of decision nodes that are indistinguishable to their assigned player.

Perfect knowledge, like that illustrated in the figure above, is a special case where each decision node is in its own information set. When players always know the history of play in the game every information set contains only one decision node. If instead some aspect of the history of play leading to an information set is unknown to the player, the player will not know exactly where in the game tree she is. This is represented by the player having more than one node in an information set. For this to make sense, the available actions must be the same at every node in an information set: otherwise, it cannot be that the player knows the game tree but is confused about where she is, because the set of available actions would rule some places out. Just because the actions are the same, however, doesn’t mean their consequences are. The future of the
game and in particular the payoffs from these actions may be entirely different depending on which node in an information set she is at. But this doesn’t allow her to choose depending on that node. She has to decide on an action without that knowledge.

Since actions can at best depend on the information set, a player’s strategy can only specify one action at each information set. To see why, imagine you are driving around out on the prairie, with an accurate map but unsure where exactly on that map you are. You arrive at a T intersection, which unfortunately is unmarked. Looking at the map you see four T intersections in what you know to be your general vicinity. The area may have a number of cross-road intersections, but you can see you are at a T. Since you can rule out the cross-roads, these four intersections are the nodes in your information set. At each of these nodes you face the choice of going left or right. Depending on which direction you choose, you might either head toward or away from your destination, so the payoffs from the choices are very different. But you cannot make your choice contingent on which intersection you are at. So whatever strategy you choose will have the same action taken at each of the T intersections. Only time will reveal the actual choice you made (or maybe not; you may never know exactly where you were).

The information sets are a crucial part of the definition of an extensive form game. In the game tree diagrams there are several ways people use to illustrate information sets. The simplest and most common is to draw a dashed line between the nodes in an information set, or to surround a set of nodes with a line, dashed or solid. So if in the example depicted above (where now I show different payoffs at the terminal node), player 2 did not know the choice of player 1 we might draw one of the following game trees:

![Game Tree Diagram]

Since player 2 cannot distinguish between the nodes in his information set, it is impossible for his strategy to call for different choices at these two nodes. At most, player 2 can choose Up or Down. Using contingent strategies is impossible because player 2 cannot fill in the information required by the “if”.

Players can of course have beliefs about where they are in an information set. These beliefs will usually affect the player’s choice in the information set, and hence which strategy they select for the game. Where exactly these beliefs come from, what restrictions on beliefs are sensible, and how players can affect each other’s beliefs are deep and fascinating questions that we will really only scratch the surface of this introductory course. We revisit this when we solve games.
In principle the uncertainty captured with an information set can be about the move of another player (as in the example above), a move of “nature” that introduces randomness, or even a previous move of the player herself. We mostly consider extensive form games with “perfect” information, where players have one node in each information set. When games have perfect information we will typically not mention explicitly the information sets, since these are identical to the decision nodes.

**The Strategic Form**

In contrast to the diagram of the extensive form, strategic form is simply a list of the game’s essential elements: players, the strategies available to each player, and the payoffs associated with each combination of strategies. This form is also called the normal form and I will use these two labels interchangeably. Little should be made of the word “normal”: its origins are mathematical, and do not imply that there is anything abnormal about the extensive form.

In the example of the previous section, players took turns, moving in sequence. This is a so-called sequential game. For many games, players choose strategies at the same time, simultaneously. These are simultaneous games. In fact, it is unimportant what the clock and calendar say about “time.” What matters is what the players know when they choose actions. If, at the time an action is chosen, a player lacks information about prior moves, strategies cannot specify actions contingent on this information. It will be as if the moves were taken simultaneously.

On the other hand, if play is sequential any given play of a game can be thought of as each player selecting one of their contingent strategies at the start of the game. As play unfolds in a sequence of moves, the contingent strategies allow a player freedom to respond to any moves made by other players. Therefore, contingent strategies can build in any response that could be made during the game. Think of strategy choice as setting a list of detailed instructions to a “proxy” player who then acts only on your instruction. (If my mother calls, tell her I’m in the bath; if my sister calls, tell her I’m out with Pat; if my father calls, get his number….). More on this in a moment.

For the case of two players, the strategic form can typically be written down as a matrix. Players are assigned to choose either rows or columns of the matrix. Strategies are represented by rows and columns, and payoffs are written as elements of the matrix. For example, the following is an example of a two player, two strategy game, a so-called 2X2 game:
Here player 1 is the so called “row player” and player 2 the “column player”. The rows and columns are labeled with the strategy names. So player 1 can choose between “up” and “down”, and player 2 can choose between “left” and “right”. The payoffs are (again by convention) listed in the order of the players, so when player 1 chooses Up and player 2 chooses Right, player 1 gets a payoff of 2 and player 2 gets a payoff of 3.

To check that you understand this notation, convince yourself of the following: player 1’s most preferred outcome follows when 1 plays Down and 2 plays Left. Player 2’s most preferred outcome follows when 1 plays Down and 2 plays Right. When player 2 chooses Left, she is indifferent to which strategy is chosen by player 1. Similarly when player 2 chooses Right, 1 is indifferent between his two strategies.

**Moving between game forms**

It is worth re-emphasizing the interchangeability of game forms. A game’s extensive form is very good for displaying how moves unfold over time; the strategic form seems naturally suited to games where all players move at the same time (simultaneous move games). However, as noted above, with contingent strategies it is possible to think of games where players move in sequence as simultaneous move games: at the start of the game, players choose a contingent strategies that incorporate a particular reaction to all potential moves of players (including themselves) over time. Since normal forms tend to be easier to analyze, translating an extensive form into a normal form is sometimes useful. (It turns out that this mapping is not unique: several different extensive forms can be translated into the same normal form. This suggests that something is lost in the translation. Whether that something is valuable has occasions some debate. What is true is that some solution concepts are harder to employ in the strategic form.)

Conversely, with information sets it is also possible to move from the strategic form of a game to an extensive form, even when players move simultaneously. (Again, this translation is in general not unique.) In fact, it is wise to interpret “moving at the same time” as an statement about limited information rather than about physical timing. If players do actually move at the same time, this explains the limited information. (Note that as our ability to observe and measure the timing of events improves it is increasingly difficult to actually move at the same time; the lack of this measurement capability is what puts more than one node in an information set.) From the perspective
of the military commander mentioned above, it really doesn’t matter if the opposing general arrays his forces before the decision of which beach to land on is taken or at exactly the same time: either way the information structure is the same, as neither general knows the choice of the other until the moves are made and fate is sealed.

It usually makes little sense to use information sets to model a truly simultaneous move game as a game tree. Nevertheless it is important for you to understand that there are no such things as “extensive form games” and “strategic form games”, though we sometimes will use this language: any game can be written in either form. Which you should choose depends on which makes more sense for the question you are addressing.

4. An Extended Example: The Battle of Salamis

In 480 BC the Persian king Xerxes was leading an invasion of Greece. (He had already pushed past the Spartan King Leonides leading the 300 hoplites at Thermopylae). The Persians were poised for what might have seemed the inevitable victory over the fractious group of Greek allies. The Peloponnesian peninsula lay behind a narrow isthmus where the Spartans and their allies had a barrier against the advancing Persians. The story of the war was recounted by Herodotus 50 years later.

By the time the Persians arrived in the neighbourhood (Attica), the Athenians had abandoned their city and relocated to the nearby island of Salamis. They were protected by the Greek fleet, lead by the Athenians but comprising ships from many cities. Xerxes faced a choice: he could either engage the Greek fleet in a naval battle or attack the land forces on the Isthmus. A victory at sea would give Xerxes’s navy a free hand to bypass the Isthmus, and complete his victory over the Greeks. A serious loss at sea would all but remove his ability to deploy his vast army that depended on the navy for its supplies.

Themistocles, an Athenian leader at Salamis, needed to maintain the support of his allies from other city states or the fortunes of Athens were unlikely to recover. There was a serious threat that allied commanders from the Peloponnesian cities would leave Salamis, dividing the fleet. Faced with wavering allies, Themistocles gave a cogent picture of the strategic situation, addressed principally to the Spartan naval commander Eurybiades. (Herodotus routinely reproduced speeches, some very many years before his time than this one.)

“It is now in your power to save Greece if you will only take my advice and stay here to fight our battle and not heed the arguments of these men by shifting our base from here to the Isthmus. Listen, and compare the arguments. If you fight a battle at the Isthmus, you will be fighting in open water, which offers us the least advantage since we have heavier
and fewer ships. Also, you will lose Salamis, Megara and Aegina – even if you are lucky at the Isthmus. Furthermore, their army will follow their fleet, and by going to the Isthmus you yourself will lead them to the Peloponnese and put all Greece at risk.

“On the other hand, if you do as I recommend, these are the advantages you will have. First, you will be fighting at close quarters with few ships against many, and if things turn out as they have before in this war we will win a great victory. At close quarters a naval battle favours us; in the open sea, it favours them. Next Salamis is saved – and that is where we have evacuated our children and our wives. And there is this further advantage, -- the one you care about most: you will be fighting for the Peloponnese just as much here as at the Isthmus, and, if you think about it sensibly, you will not be leading the enemy into the Peloponnese.

“If it turns out as I hope, and our ships are victorious, the barbarians won’t advance on you at the Isthmus or indeed go any farther in than Attica. They will retreat in disarray, and we will benefit from having held on to Megara, Aegina, and Salamis, where the oracle says that we are destined to have the upper hand. In general, things go well when people make sound plans; when their plans are flawed, even god had no use for their mere mortal opinions.”

Questions:
1. Formalize the situation that Themistocles describes as an Extensive Form game.
2. Re-write the game in Normal Form.

Suggested solutions:
1. To formalize this as a game, we need to identify players, strategies, payoffs and information.

Players: Since the strategic situation, at this level, is between the armies of Greece and Persia, we can treat these as unities: the players are Greece and Persia.

Strategies: The question Themistocles is addressing is whether Athens’ allies should move to the Isthmus or stay and fight at Salamis. The Persians have a more complex problem, deciding whether to attack now, move to the Isthmus, or wait out the Greeks, blockading them, divide their forces, etc. One simple and clean interpretation is that

---


4 Other debates presented by Herotodus outlines some of these options (notably a passionate speech by Antonina of Halicarnassus, the ruler and admiral in charge of the forces from Herotodus’s hometown.
the Persians must choose between fighting now at Salamis or later at the Isthmus. (There are others.)

The actions are then:

Greeks: remain United; Divide the navies
Persians: Salamis; Isthmus

Information: The speech by Themistocles is predicated on the Greek’s option to move first. That is, they can choose to stay united or divide the navies before the Persian’s decide on approaching Salamis or making for the Isthmus. It isn’t clear from the passage cited above exactly what the Persians know about the Greek strategy. In fact, Themistocles was actively attempting to deceive the Persians about the unity of the Greeks: simultaneously with his urging unity, he was sending a false message of disunity to Xerxes. For the benefit of the example, I will consider both possibilities: A1 Xerxes doesn’t observe the Greek’s choice, and A2 Xerxes does observe the choice.

Under A1, both sides have two strategies, identical to their actions stated above. Under A2, the Greeks continue to have two strategies. But now the Persian’s can make their choice contingent on the Greek strategy, so they have four strategies:
   i. (Salamis|United) & (Salamis|Divided);
   ii. (Salamis|United) & (Isthmus|Divided)
   iii. (Isthmus|United) & (Salamis|Divided)
   iv. (Isthmus|United) & (Isthmus|Divided)

Recall that the vertical line “|” is read as “conditional on”.

Payoffs: Not surprisingly, Themistocles does not assign numerical values to the outcomes. But he does distinguish somewhat between a simple victory and a great victory. Presumably one side’s gain is the others loss, or at least approximately so. Themistocles also doesn’t assign values to every possible outcome, at least not explicitly. For now, I will suggest some payoffs for the outcomes Themistocles does mention.

Assuming that one’s side gain is the other’s loss means the payoffs sum to a constant, which, due to the flexibility of assigning specific numbers, can be taken to be zero. So called “zero-sum games” are strictly competitive. These are a special case with no room for cooperation. Military conflict is typically taken to be zero-sum. But in many instances this is incorrect, hence the seemingly oxymoronic “rules of warfare”. In fact, most social situations have some room for mutually beneficial strategy combinations.
The extensive form: the only difference between the extensive forms for the two variants of the Battle of Salamis is the information set for the Persians. Under A1, the Persians have two information sets, which allow them to distinguish between the situations following the Greeks’ choice of remaining united or dividing. Under A2, they cannot distinguish these situations, so they have only one information set. It is as if they move at the same time.

The Battle of Salamis A1.

The Battle of Salamis A2.

The Normal Form: Assigning the rows to the Greeks and the columns to the Persians, the normal forms for the two versions of the game are as follows:
Recall that the notation $(S|U, S|D)$ in the normal form under $A1$ indicates the Persian strategy to choose Salamis conditional on the Greeks staying united, and choose Salamis conditional on the Greeks dividing their forces. In other words, always choose Salamis.

The timing is hidden in the strategic form: this simply lists the strategies available to the players. The whole game can be played by each player simultaneously choosing a strategy. Under $A2$ neither player’s strategies are contingent, while under $A1$, when the Persians observe the choice of the Greeks, their strategies can condition on this choice. But as you see, this is more a statement about what is known when moves are chosen than “who moves first”. (It would of course seem odd to say that the Persian’s “move” first; but they could commit to a contingent strategy at the same time as the Greeks.)

5. Practice Questions

1. In the fall of 2008 financial markets were paralyzed and many people feared that the world was destined for a prolonged slump, perhaps even rivaling the Great Depression of the 1930s. To prevent this, governments from around the world met to plan how to stimulate the economy thought running deficits.

Assume for simplicity that there are only two countries in the world, and that they are identical in population and the size of their economies. Assume further that for the demand stimulus to be effective, both countries must run deficits. In
this case, both countries will be made better off. If only one country runs a deficit, the world economy will still enter a depression. The country with the “stimulus” deficit will be worse off than if it has tried to balance its budget. The country that did try to balance its budget will be unaffected by the other country’s stimulus.

(a) Write the normal form for this simultaneous move game.
(b) Which, if any, of the “classic” games is this an instance of?

2. Traffic is very slow this morning northbound on Shelbourne Street. Somehow, someone has dropped a mattress in the right lane, and cars must merge into the left lane to get past it. The mattress doesn’t weigh much so any driver could feasibly pull over and move it out of the way. This is obvious to everyone when they get to the obstruction, but by then pulling over just increases to their delay.

Assume that there are only two drivers coming up the right lane. Merging around the mattress adds 5 minutes to the commute time, but stopping to move the mattress adds 8 minutes to the commute. If both drivers stop, both are delayed by 8 minutes. If only one driver stops to move the mattress, that driver is delayed by 8 minutes but the other is not delayed at all.

(a) Assume that both drivers must decide in advance whether to stop and move the mattress or to merge around it and continue on. Write down the normal form for this simultaneous move game.

(b) Now assume that the players arrive at the mattress sequentially: player 1 gets there first, followed by player 2. Represent this as an extensive form game. List the strategies available to player 2.

3. Simon loves to pose puzzles for his dinner guests. One night he had three guests Alberta, Brittney and Claudia. After dinner, he had everyone close their eyes and then placed blue hats on Alberta and Brittney and a red hat on Claudia. Everyone could see the other two people’s hats, but not their own. Simon said that he would have chocolate cake for anyone who could answer his question about the colour of the hats, but guessing wrong would mean no cake and instead having to do the dishes. He told the guests that the hats were either blue or red. He asked if anyone could tell the colour of their hat. Everyone said no. Then he decided to give a hint: he said at least one of you is wearing a blue hat. Then he asked again. Still no one answered. Then he asked a third time. This time both Alberta and Brittney said they were wearing a blue hat. After this, Claudia said she was wearing a red hat.

How did the guests figure this out? How did Simon’s hint help?
4. An old drinking house game is played on a bar with matches. In the typical game, the game starts with 21 matches laid out in a row. A coin toss determines who plays first. At each turn players may choose to remove either 1 or 2 matches from the row. The player who removes the final match loses.

Write down this game as an extensive form. For simplicity assume that there are 5 instead of 21 matches. Ignore the coin toss and simply assign one player as player 1.
Chapter 2. Equilibrium

This lecture introduces the non-cooperative solution concepts used to analyze what happens in a game played by rational players independently choosing strategies. A solution is a set of strategies, one for each player. Solution concepts are criteria for picking particular strategies from among all the possible combinations of strategies players can feasibly choose. A specific solution is often referred to as an equilibrium of the game, and solution concepts are also called equilibrium concepts. Solutions are equilibria in the sense that there is no force internal to the model that would lead an individual player to change their choice of strategy away from an equilibrium\(^5\). Even though an equilibrium is a “rest point” for strategy choices, strategies themselves can allow for changing behaviour, and equilibrium outcomes can change over time.

Depending on, among other things, the solution concept used, a game may have one or many equilibriums (or equilibria). From the point of view of predicting play, the fewer are the equilibria consistent with a solution concept, the better. After a couple of more definitions, we develop the concepts in the context of some very widely discussed games, which I refer to as “classic games.” In this chapter, all games are presented in the strategic form. Extensive form games are solved in Lectures 3 and 4.

As described in lecture 1, our approach is to divide our analysis in to a “specification step” that formalizes a strategic situation as a game, specifying players, information, strategies and payoffs, and a “solution step” that applies an equilibrium concept to predict what will happen when the game is played. Lecture 1 provided tools and concepts to formalize situations as games. Once a game is correctly formulated, solving it is often relatively straightforward. When you arrive at a result that is confusing, unfamiliar or just plain “wrong,” attacking the logic of the solution method is usually less productive than re-examining the specification of the game. Far too often an inappropriate (or just poorly modeled) game with absurd conclusions is used to argue that “traditional analysis” is wrong. The sensible reaction when a game misses the mark is to reconsider your formulation of the game, and then reapply a logically valid solution concept. This process of revision that forces you to carefully consider your beliefs and assumptions is often the most useful consequence of using game theory.

That said, the two-step methodology is not absolute. In many cases there will be a number of potential solution concepts available for a game. An ideal solution concept would deliver one single prediction for every well formed game. (That is, for any well formed game, a prediction would always exist, and always be unique.) This solution

---

\(^5\) Remember, we are assuming that strategies are chosen by players acting independently. This is the essence of “non-cooperation.” Later, we will briefly consider “cooperative” game theory which assumes that players can make binding agreements to choose strategies collectively.
concept could then always be used, and would always tell us exactly what the players will do. Unfortunately no such solution concept exists. Instead a variety of solution concepts have been developed, which are more or less applicable to different games (i.e. different assumptions about players, actions, information and preferences). Here, and in the coming lectures, we encounter a number of the most basic and useful solution concepts, but we will not exhaust the possibilities by any means. Some of the most interesting (and difficult to understand and apply) solutions concepts involve assumptions about how players form beliefs in games without complete and perfect information. This is a class of games we will largely ignore.

Solution concepts can be thought of as “strong” or “weak”. Strong concepts impose strict conditions that must be satisfied by a set of strategies to qualify as an equilibrium. This is good in that it limits the potential number of equilibria predicted by the concept, but if too strong, it may be that no set of strategies satisfy it. This isn’t very helpful. We start with a very strong solution concept, Dominant Strategy Equilibrium. This is convincing when it applies, but because the requirements are very strict, it is only useful in a small set of games. Weak concepts impose less stringent conditions and include more sets of strategies as solutions. Weak concepts are more likely to admit solutions, but in the limit a very weak solution might let “anything go” and be useless for prediction. As noted, the unattainable ideal is a solution that applies universally and always selects a unique equilibrium.

The cornerstone solution concept of non-cooperative game theory is Nash Equilibrium. Weaker than Dominant Strategy Equilibrium, Nash equilibrium predicts a solution for essentially all well defined games. The inevitable downside of Nash Equilibrium’s generality is that it admits more than one solution to many games. In response, game theorists have developed a wide range of additional restrictions that can be added to Nash to select among – or refine -- the multiple equilibria. Typically a refinement is motivated by some specific characteristic of the game, and are best applies in a limited set of games. Some particularly interesting refinements are based on the information available in dynamic games. These refinements are easier to explore in the extensive form, and we will return to this subject in the next lecture.

Key terms: Dominant strategy; best response; cooperative solution; Nash equilibrium; focal point.

---

6 Notice that this adulterates our conceptual division between the specification and the solution of a game.
1. Best responses

A rational player in a non-cooperative game examines the set of available strategies and chooses the one that leads to the most preferred outcome. Since outcomes depend on the strategy choices of other players, selecting a good strategy required players to form beliefs about the actions of other players. A player’s best response is a strategy that produces the highest payoff for that player, given the player’s belief about the strategies chosen by the other players in the game. The assumption that every player will choose a best response is fundamental to rational choice game theory. Consider Example 1, the two player game illustrated below:

```
<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>2,4</td>
<td>2,3</td>
</tr>
<tr>
<td>Down</td>
<td>3,4</td>
<td>2,5</td>
</tr>
</tbody>
</table>
```

A best response for player 1 is a strategy choice that is best given a belief about the strategy chosen by player 2. In this simple game, player 1 has two\(^7\) possible beliefs about player 2’s strategy choice: that player 2 is choosing Left, and that player 2 is choosing Right. For each of these potential strategy choices by player 2, we calculate player 1’s best response by asking which of player 1’s strategies leads to the highest payoff, given that choice by 2. In the game illustrated above, when player 1 believes that player 2 is choosing Left, Player 1 can choose Up, for a payoff of 2, or Down, for a payoff of 3. Because Down yields the higher payoff when player 2 plays Left, we say that Down is Player 1’s best response to Left. Similarly, player 2 had best responses to the strategy choices by player 1. For example, Left is 2’s best response to Up.

Exercise. Use the normal form game in Example 1 to work out: player 1’s best response to Left; player 1’s best response to Right; player 2’s best response to Up; and player 2’s best response to Down.

Nothing guarantees that a player has a unique best response to an opponent’s choice. For a given strategy choice by her opponents, more than one of her strategies might yield the same (highest) payoff. Then all of these equally valuable strategies are best responses, and we say that 1’s best response is not unique. In the example above both Up and Down are best responses for player 1 to player 2’s strategy Right. So the mere assumption that players choose best responses does not necessarily isolate a specific choice.

---

\(^7\) Ignoring, for the moment, the possibility that player 2 randomizes her choice between Left and Right. We consider this possibility later when we examine “mixed strategies”.

©David Scoones 2007-17
It is less obvious that sometimes best responses do not exist: perhaps no strategy available to a player is best against some strategy choice by other players. As you saw from the exercise just above, this isn’t true in Example 1, and won’t be true in similarly “well-formed” games. To guarantee that a best response always exists, the game must satisfy some technical assumptions. Stating these assumptions precisely requires formal mathematics, essentially restricting the properties of the set of strategies each player can choose among. I intend to present only well-defined games in which players do have best responses, so implicitly satisfy these requirements. A more careful examination of these technical niceties is left to your future studies.  

2. Dominant Strategy Equilibrium

Consider a player with at least two strategies, A and B. When strategy A has a higher payoff than B for every strategy that can possibly be chosen by her opponent, we say that “strategy A dominates strategy B”. Equivalently, we can say that “B is dominated by A”. Since A always produces a better outcome, there is no reason to use B when A is available. When one strategy is dominates every other strategy we say that strategy is a dominant strategy. If our example player has only strategies A and B, then A is dominant. When a player has a dominant strategy, her beliefs about the choices of other players do not matter because the dominant strategy is best for any beliefs. Rational players will always play a dominant strategy when they have one, as it is always a best response to any and all strategies of the other players.

When a player has a dominant strategy the only reasonable belief for other players is that she will choose her dominant strategy. When every player has a dominant strategy, the solution to a game is straightforward and convincing. Since no rational player will

---

8 To get an idea of what can go wrong, imagine a game where you are required to name a “real number” strictly less than $10, following which I pay you that number in money. Whatever number you choose, there is always a larger one that is still less than $10 – if you said $9.9999, you could have instead said $9.99999, etc. Assuming you value money, for any amount you name, some “feasible” alternative will be better, so you can never name a truly “best” response. Our technical restrictions would rule this situation out, saying that the “strategy set” (an open set of real numbers) is not well-defined. The problem disappears if strategies are restricted to being expressed in units no smaller than a penny. This might seem a reasonable and realistic restriction. Then you have a best response: $9.99. This is the kind of restriction we will impose without comment.
ever choose a dominated strategy, all players will play their dominant strategy. The list of these dominant strategies, one for each player, is called a dominant strategy equilibrium.

Some very interesting and widely studied games have a dominant strategy equilibrium. Nevertheless, it is a special case, and in this sense the concept is “too strong”. While clear and convincing when it exists, in part because it sets aside entirely the question of beliefs about other players’ actions, it generally doesn’t exist.

3. Nash Equilibrium

Nash Equilibrium is a much more widely applicable solution concept, requiring simply that all players choose “mutual” best responses. Mutual means your strategy is a best response to mine, while at the same time, mine is also a best response to yours. A Nash Equilibrium (NE) is a list of strategies such that every player chooses a best response to the strategies chosen by the other players.

One straightforward way to find Nash Equilibria is to test each of the possible strategy combinations, and see if any particular combination constitutes mutual best responses. (This can be impractical if there are many players or many strategies.) If any player’s strategy is not a best response to that of the others, the strategy combination isn’t a NE. In describing this process, it is common to say that we are testing to see if any player would wish to “deviate” from the candidate equilibrium, by choosing an alternative strategy.

It is crucial you remember that these “deviations” are unilateral. Players do not coordinate or cooperate to find better alternatives; deviations are tested assuming that all other players stick to the strategies in the candidate equilibrium. This is why we call it non-cooperative game theory. We are not asking whether collections of players can improve their outcomes by a coordinated move to a different set of strategies. If cooperation emerges in equilibrium, and you will see that it sometimes does, it is the result of individual actions.

Let’s apply this method to Example 1:

```
<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Up</td>
<td>2,4</td>
</tr>
<tr>
<td>Down</td>
<td>3,4</td>
</tr>
</tbody>
</table>
```

First, test the candidate equilibrium strategy combination (Up, Left). Is this a Nash Equilibrium? If player 1 chooses Up, Left is a best response by player 2. So far so good,
but this is only half of what we need, as both players must be choosing best responses. We must confirm that when player 2 chooses Left, Up a best response for player 1. This is not true: the best response to Left is Down, not Up. That is, if player 1 believes player 2 is choosing Left, he would prefer to “deviate” from the candidate equilibrium and play Down. Hence, Up and Left are not mutual best responses, and (Up, Left) is not a Nash Equilibrium.

In games like Example 1 with two players each of whom have two strategies (so-called 2X2 games) it doesn’t take long to do this for the four possible combinations of strategies. In Example 1, this demonstrates that player 2 will deviate from (Down, Left) and from (Up, Right), hence neither of these is a NE. Make sure you understand why this is so. The only remaining candidate equilibrium is (Down, Right). Is this a NE? Yes: Right is a best response to Down, and Down is a best response to Right. These are mutual best responses, neither player has the incentive to unilaterally deviate, and (Down, Right) is a Nash Equilibrium.

One way to think about NE is that players form beliefs about the strategies of other players and play best responses to these beliefs. In a NE these beliefs are correct, and will be validated by the opponents’ strategy choices. There are no surprises, and no regrets. In Example 1, player 1 has no strong reason to play Down. In fact, given the “equilibrium belief” that player 2 plays Right, player 1 is indifferent between Up and Down – either choice yields a payoff of 2. Nash Equilibrium does not require players to have strong reasons to make the choice they do; instead, it requires that they do not have any reason to make an alternate choice. However, in equilibrium Right is a best response for player 2, and this requires player 1 to play Down, despite having no strong reason to do so. Even though player 1 is indifferent between his choices in equilibrium, the equilibrium requires him to play Down.

In Example 1 there is only one Nash Equilibrium, but nothing in the definition of NE requires this to be true. When it is true, we say that the NE is unique; when it’s not we say there are multiple equilibria. Many interesting games have multiple Nash equilibria, and additional considerations are needed to select an outcome appropriate for the circumstances modeled by the game. These additional considerations restrict the set of Nash Equilibria, and are sometimes called refinements (as they reduce or “refine” the set of equilibria). These might be purely logical considerations applying to all similar games (we’ll see an example in the next lecture). Or they might derive from some
aspect of the situation studied that is not formally included in the specification of the game. For example, when players are “symmetric,” with the same preferences and strategies to choose from, it might make sense to focus on a symmetric equilibrium, where all players choose the same strategy. Aspects like symmetry may make one NE salient or a focal point for player beliefs. Thomas Schelling was an early expositor of this line of thought, and some people refer to these salient equilibria as Schelling Points.

In some games the fact that a particular combination of strategies forms a Nash Equilibrium might in itself provide a reason for players to choose those strategies. This is usually explained with the claim that, if all players have access to the same game theory handbook, they can compute the NE: since it uses best responses, no player has the incentive to not follow the handbook’s advice. This argument is particularly compelling when the equilibrium is unique.

Finally, it should be apparent that a dominant strategy equilibrium is always a NE. In a dominant strategy equilibrium, every player chooses a strategy that is a best response to any choice of strategy by the other players, so their equilibrium choice must be a best response to the strategy actually chosen by the other players. The test for a NE will always find a dominant strategy equilibrium should one exist. This is another sense in which dominant strategy equilibrium is a stronger equilibrium concept: all dominant strategy equilibria are Nash, but not all Nash equilibria are dominant strategy equilibria.

---

John Nash formally defined Nash Equilibrium and proved that very generally there exists at least one NE any well defined game. His proof is the reason the concept bears his name, and the fact of its general applicability the reason for its widespread use. Nevertheless, you can easily construct two player, two strategy games where none of the four strategy combinations constitutes a NE. Nash’s proof of existence allows “pure strategies” (like Up or Right) to be chosen randomly. These randomisations are called “mixed strategies.” Mixed strategies are both technically and intuitively difficult, but useful and interesting; we take them up in a later lecture.

4. Classic Games

This section applies the concept of Nash Equilibrium to set of games that comprise an essential part of the game theorist’s toolkit. Some of these games are also referenced in the media and other public discussions of strategic situations. Each “classic” game is designed to highlight a particular feature of some strategic interaction, and over time the attached story has come to serve as a shorthand reference for a particular type of social situation. The stories underlying these games might be thought of as fables; like fables, they can be a powerful and memorable way so isolate the essential aspects of an otherwise complex situation. However, the value entirely depends on getting the story straight! Some of these classic games are so beguiling that people start to see them even where they are inappropriate, and rather than clarifying they distort
understanding. You should become familiar with the names and stories, and learn to apply them thoughtfully and critically.

(a) The most famous of all the classic games is certainly the Prisoner’s Dilemma. Among other things, this is a stock scene in TV crime dramas. Two criminals — we’ll call them Albert and Brian — are captured then separated by the police and offered deals if they confess and provide evidence on their partner in crime. The normal form is

<table>
<thead>
<tr>
<th></th>
<th>Confess</th>
<th>Don’t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>-10, -10</td>
<td>0, -20</td>
</tr>
<tr>
<td>Don’t</td>
<td>-20, 0</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>

Albert is the row player, whose payoff is listed first in the payoff matrix, and Brian is the column players, whose payoff is listed second. The payoffs are roughly chosen to represent the number of years spent away from life outside of jail. Larger in absolute value negative numbers are smaller payoffs, and represent less preferred outcomes. In the story, when both players confess, the crown does not require testimony, but because Albert and Brian take responsibility, the sentencing is relatively lenient. When neither confesses, the crown cannot make the main charge stick, and so both are convicted on some relatively minor alternative charge. When only one confesses, he goes free, and the other serves a long sentence. The numbers in the various versions of the prisoner’s dilemma vary a bit, but the strategic structure is always as in the game above.

To compute a Nash Equilibrium, we will work through the best responses. This is made faster because the game is symmetric, so we can take either player as representative (Whatever is true for Albert will be true for Brian, just by swapping names.) Doing so, you should see that the best response to Confess is Confess, and the best response to Don’t Confess is Confess. Each player has a dominant strategy to Confess, so the game has a dominant strategy equilibrium! You can easily see that this gives each player his third best payoff.

The enduring popularity of the prisoner’s dilemma is due to it predicting that the dominant (unambiguously best) individual choices lead players to be collectively worse off than they could be if they could somehow coordinate their choices. If players could commit to strategies before being arrested, they could agree to not confess. But that is...

---

9 As you recall, this is also a Nash Equilibrium: if a given strategy is a best response to all of the strategies of other players, then it must be a best response to the strategy chosen by the other player.
a big if. In the non-cooperative equilibrium they get 10 years each, not 1 year only on the lesser charge they get with the strategies (Don’t Confess, Don’t Confess).

The strategies (Don’t Confess, Don’t Confess) are said to be the game’s cooperative solution or sometimes cooperative equilibrium. But it isn’t a Nash equilibrium. At the cooperative solution it is impossible to make one player better off without making the other worse off. Outcomes with this property are called Pareto Efficient (or Pareto Optimal). The fundamentally important feature of the Prisoner’s Dilemma is that the dominant strategy equilibrium is not Pareto efficient.

Players would both prefer to arrive at the cooperative solution, but that alone doesn’t make it achievable. The Prisoner’s Dilemma provides no means to enforce a cooperative agreement. Since confessing is a dominant strategy, whether or not a player thought the other was going to cooperate, he would be better off confessing. The cooperative “equilibrium” is not an equilibrium in the non-cooperative game. To make it an equilibrium, we need to change the game, by adding so means of commitment. One way not to do this is to “fix” the story by saying that each player would care about the other, and resist making choices that damage the other’s interests. It is fine to account for such “other regarding” preferences, but you do this by modifying the player’s own payoffs. If this change removes the dominance structure, so it is no longer the prisoner’s dilemma.

Since the Prisoner’s Dilemma was first studied in the 1950s, many social situations have been cast in its form. The problem of individual optimization leading to inferior social is widespread and costly, so understanding when and why this happens is of obvious interest. Because of its pervasiveness, this situation is sometimes given the more general name of a social dilemma10. The problem does not go away when the number of players increases, but the problem of enforcing cooperation becomes more complex.

(b) Closely related to the prisoner’s dilemma is Stag-Hunt. This story is also about the difficulty of reconciling individual choice and collective values, and was described long before the prisoner’s dilemma. (Rousseau mentions the problem, placed in the setting of hunters pursuing deer; this is the source of the modern name. Hume also mentions the basic situation.)

In the stag hunt, two (or more) hunters are in the field pursuing a deer (stag). To be successful both must concentrate on the hunt, but this isn’t guaranteed. Each hunter can choose instead to abandon the stag hunt and trap a rabbit (hare). Half a stag is better than a hare, but hare hunting does not require cooperation.

The strategic form for this game might be

---

10 Sometimes this more general label is reserved for games with continuous strategy spaces. Further classification depends on whether players choose inefficiently to do too much or too little of some activity than they should to maximize social value.
In this standard version, the payoff for hunting hare is independent of the strategy chosen by the other player (you should confirm it is 1 in each case). Slight variants of the Stag Hunt exist, and sometimes the payoff for Hare is slightly greater when the other player hunts Stag than when both go after Hare. That variant, sometimes called an Assurance Game, is much studied in international relations, among other places. Provided the payoff doesn’t increase much, the basic strategic situation is unaffected (something you should convince yourself of).

By now you should be able to work out following results for this game. Neither player has a dominant strategy, so the game has no dominant strategy equilibrium. However, Stag Hunt does have two Nash Equilibria. For both players, the best response to Stag is Stag, and the best response to Hare is Hare. The Pareto optimal outcome, where a stag is caught, does occur in one of the Nash Equilibria (Stag, Stag). But the inefficient outcome where both catch a hare (Hare, Hare) is also an equilibrium.

Since the Nash Equilibrium is not unique, we need further considerations to predict which of these strategy combinations will be chosen. Both NE are symmetric, so that doesn’t help. Are there other focal points? What might make either (Stag, Stag) or (Hare, Hare) salient in this game? History or custom suggest themselves, and may play a role in many real-life stag hunts. But this just begs the question of how players come to cooperate, and we need a new model that explains how the custom arose.

On the other hand, the fact that (Stag, Stag) is better for both players might, in itself, make this equilibrium salient. That is, (Stag, Stag) is the so called payoff dominant equilibrium. Because it is better, this equilibrium might attract attention, and be the salient thing to coordinate on. (Notice that, even though this one is, payoff dominant equilibria don’t in general have to be Pareto efficient; they need only be better for both players than the other equilibria not better than all feasible outcomes.)

However, this isn’t the only possible story you might tell to decide what is salient. Suppose one player assume that (Stag, Stag) is being played (perhaps because it is payoff dominant), but the other player believes it is (Hare, Hare) that is chosen. Since (Hare, Hare) is an equilibrium, this belief might be true in some equilibrium so can’t be totally ruled out. (Perhaps that player grew up in a place with low levels of cooperation.)

---

11 Note that this isn’t the same thing as a dominant strategy equilibrium.
This breakdown of coordination leaves the Stag player with a payoff of 0. In contrast, playing Hare provides insurance against this confusion, always ensuring a payoff of 1. A player who chooses to hunt hare has is no worries that the other player choosing the “wrong equilibrium.” Ironically, this very reasoning provides a justification for the belief that (Hare, Hare) might be the equilibrium! Because it is safer, hare hunting is called the risk dominant equilibrium.

Not surprisingly, Stag Hunt game is a useful tool for thinking about the role of trust in social situations. When players know each other and can rely of social norms, it seems sensible that they can coordinate on the payoff dominant equilibrium. In the absence of trust, it may make sense to be cautious, choose the risk dominant equilibrium, and endure an outcome that is collectively worse. Sadly, suspicion breeds suspicion. Like the prisoner’s dilemma, these problems remain when there are many players.

(c) Coordination games are games where players benefit from taking similar actions. A simple pure coordination game is played every time drivers take to the road. Assume there are only two drivers and each must choose to drive on the Left or the Right. As long as each chooses the same side, it doesn’t matter which is chosen; the bad outcome occurs when each driver chooses a different side (and hence end up on the same piece of pavement). A normal form might be

<table>
<thead>
<tr>
<th></th>
<th>Driver 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Driver 1</td>
<td>Left</td>
<td>Right</td>
</tr>
<tr>
<td>Left</td>
<td>2,2</td>
<td>0,0</td>
</tr>
<tr>
<td>Right</td>
<td>0,0</td>
<td>2,2</td>
</tr>
</tbody>
</table>

Work out the best responses for each player and find the Nash Equilibria. You will see that like Stag Hunt there are two NE, but in this game neither is either payoff or risk dominant. Even though this is a symmetric game, symmetry doesn’t help either\(^{12}\). Abstract reasoning doesn’t lead to an obvious focal point. This is actually comforting because both equilibria exist in various parts of the world. Local tradition marks one equilibrium as appropriate. To make the point clear, jurisdictions have laws to “enforce” the equilibrium. Do you think these laws are frequently challenged?

Many games involve some element of coordination, and others are “anti-coordination” games in which the best outcome required players to take different actions. In fact, these are more closely related thank it might appear on the surface. We can typically rename strategies in a way that make anti-coordination games into coordination games. For example, the driving game can be made an anti-coordination game by relabeling strategies as North and South (or East and West). Then the equilibria would require one driver to choose North and the other South.

\(^{12}\) It symmetric because other than the labels 1 and 2, there is nothing to distinguish the players.
(d). A coordination game with a twist goes under the lugubrious name of the **Battle of the Sexes** (or BoS. Other names exist, but this one is the most common.) The story here is that a couple only have fun when they go out together, but each has a different idea of what makes for the better entertainment. The best outcome for a player is they meet at his or her preferred event. The worst is when they don’t meet. To minimise the sexist overtones, we will follow Osborne and Rubinstein in relabeling this game as Bach or Stravinsky: The normal form is

<table>
<thead>
<tr>
<th></th>
<th>Bach</th>
<th>Stravinsky</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sam</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Pat</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

The players must simultaneously decide which concert to attend. Each prefers to go to the concert chosen by the other to attending a concert alone, but Sam prefers they both attend Bach while Pat prefers they attend Stravinsky. Again you will find two NE, but now the players disagree on which is best. What might make one of these a focal equilibrium?

(e) Next to the prisoner’s dilemma, the most often cited of these allegorical games is probably **Chicken**. Bertrand Russell first popularised this game as a metaphor for dysfunctional social relations. The standard story is sometimes said to derive from a classic James Dean movie about alienated teenagers speeding their cars towards a cliff. The first to “swerve” loses face, but, on the other hand, not swerving is fatal. The game is easier to write down when it’s reframed to have the lads speeding toward each other, and crashing unless at least one swerves. Both drivers most preferred outcome is when he does not swerve but the other does; next best is both swerving (limiting the loss of face); third best is swerving when the other does not, losing face but saving life and limb; finally, the worst outcome is when neither swerves, and the crash ensues. A normal form for the game is

<table>
<thead>
<tr>
<th></th>
<th>Swerve</th>
<th>Don’t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Driver 1</td>
<td>2,-2</td>
<td>-5,-5</td>
</tr>
<tr>
<td>Driver 2</td>
<td>0,0</td>
<td>-2,-2</td>
</tr>
</tbody>
</table>

Again you should be able to work out that this game has two NE (and no dominate strategies). This is an “anti-coordination” game as the two equilibria call for different actions by the two players. What if anything makes one of these salient is hard to
imagine, but tattoos come to mind. Chicken in other settings is sometimes referred to as Hawk-Dove. We’ll encounter it again with this label when we discuss evolutionary games.

(f.) Zero-sum games are games in which one player’s gain is always another player’s loss. Many games played for fun and most professional sports are zero sum if the only thing that matters is winning or losing. Zero-sum games are sometimes called purely competitive as there is no role whatsoever for cooperation. Military engagements are often considered zero-sum, with a single victor, and this accounts for their extensive study in the earliest applications of game theory. On the other hand, some people seem to see zero sum games everywhere they look, and view society as a conflict over resources.

When one player’s gain is always another’s loss, it is possible to choose payoff numbers in a way that for any given outcome the payoffs sum to zero: hence the name. The classic example is Matching Pennies. Each player simultaneously places a penny on a table, choosing whether the upper side is a Head or a Tail. When both players choose the same side, player 1 wins and collects player 2’s penny. When the sides do not match, player 2 wins. The normal form is

<table>
<thead>
<tr>
<th></th>
<th>Head</th>
<th>Tail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
<tr>
<td>Tail</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
</tbody>
</table>

For player 1, the best response to Heads is Heads and to Tails is Tails. For player 2, the best response to Heads is Tails and to Tails is Heads. So none of the four combinations of “pure strategies” (H,H), (H,T), (T,H) or (T,T) is a Nash Equilibrium. To solve this game we need to introduce the concept of mixed strategies (in Lecture 6).

5. Alternative Solution Concepts

   Iterated Elimination of Dominated Strategies

In the prisoner’s dilemma, each player has a dominant strategy (Confess). This makes their choice easy, so easy that players do not need to anticipate the choices of their opponents. But if a player were to think about it, he would expect that they would also play the dominant strategy. He can act as if the opponents didn’t even have the option to play the dominated strategy.
More generally, predictions of player choices are not affected when we “eliminate” (strictly) dominated strategies from their set of choices. Moreover, by removing dominated strategies from one player, we reduce the set of possible outcomes, and this can mean that some strategies in other players are now dominated in the reduced game. Carrying on, these “new” dominated strategies can be eliminated, removing another batch of outcomes. The can imply other strategies dominated, and so on. Because no player will ever play a strictly dominated strategy in Nash Equilibrium, this process will never delete any NE. Sometimes the process will converge, leaving a single strategy for each player; if so, those strategies constitute a NE. (Iterated elimination of weakly dominated strategies may remove some NE.) When it works, this process is a simple and effective solution method. See the example below:

For player 1, Middle is strictly dominated by Up. Eliminating this option from consideration, player 2 finds Left is dominated by Right (even though it would not have been had there been any chance player 1 would choose Middle). Deleting Left, in turn means that player 1 will never play Down. Thus Up is the only reasonable expectation for 1’s choice. Given this, player 2 will select Right. It is tedious but straightforward to verify that (Up, Right) is the only NE.

**Cooperative game theory**

In the prisoner’s dilemma I referred to (Don’t Confess, Don’t Confess) as the “cooperative solution”, but it isn’t really a solution at all in the non-cooperative game. In non-cooperative game theory players are assumed to act independently, and cannot commit to strategies that make them worse off. But in other games, (for example, stag hunt) players can sustain cooperation. This raises the issue of sharing the benefit from cooperation. To study this sharing, it can be convenient to simply assume players can commit to playing “cooperative strategies” that ensure the largest collective payoff, and then ask how, then, would players divide the benefit from cooperation. Cooperative game theory does this. The emphasis is on the structure of cooperative agreements, rather than the individual strategy choices that underlie cooperation. As such it is a distinct branch of game theory.

Cooperative game theory is complementary to non-cooperative game theory. Many game theorists consider cooperative game theory solutions as less fundamental than Nash Equilibrium, as these solutions assume behaviour that might not be in the interest
of players. This assumption implies that some un-modelled mechanism exists to enforce cooperation; this means that a fully specified non-cooperative game should exist to model this mechanism and produce cooperation in equilibrium. Be that as it may, it is clear that many situations exist in society where cooperation is sustained. It is useful and interesting to examine the structure of cooperative agreements without all the machinery of a fully specified non-cooperative game getting in the way. Cooperative game theory allows for discussion of issues like “fairness.” We will tour cooperative game theory solutions in Lecture 5.

Minimax

Many early games were zero-sum, used to model warfare, for example, or sports and parlour games, where this simplifying assumption was seen as appropriate (if never entirely accurate). Before Nash, the most important solution concept in game theory was the Minimax solution. Minimax strategies minimize the maximum payoff to the opposing player. When players’ payoffs sum to zero (one player’s gain is the other’s loss), this is equivalent to maximising the minimum payoff that your opponent can force you to receive. One motivation for this concept imagines that an opponent will somehow discover your strategy in time for them to respond. When payoffs are zero-sum it makes sense for them to choose the strategy that, given your choice, yields them the highest possible payoff (and so you the lowest, consistent with your strategy). Anticipating this, you should assess your strategies by the lowest payoff for your opponent (and so the highest for you) when they pick the strategy that makes their payoff as high as possible conditional on your choice. That is, you should minimize their maximum payoff: hence the name Minimax.

The Minimax solution has many interesting mathematical properties. The reasoning behind it may seem a bit paranoid – it was developed by Dr. Strangelove, after all – but paranoia might make sense in zero-sum games. But it makes very little sense for games that are not zero-sum. Because of this (and in the interest of time) we will not consider the minimax solution further in this course. I mention it here only because you are certain to encounter this idea eventually. It will not be covered by any exam.

Practice Questions:

1. Consider the following “study game”. Isis and Janis are equally good students who are taking a class together. Grades in this class are grades are determined competitively: if both perform equally well, they will both get an A; if one out performs the other, the one who does better will get an A while the other will get a B. Performance depends on their effort studying. Both Isis and Janis prefer an A to a B, and prefer being the top student, but studying is costly.

(a) Write down the normal form for this game.
(b) Compute the best responses for Isis to the strategies chosen by Janis. Does she have a dominant strategy?

2. Avon and Omar are both quite willing to sell drugs for a living. Avon typically buys his at a hefty discount directly from an importer, while Omar prefers to increase profits further still by stealing his supplies from Avon. Avon is planning a transfer and has two choices: by the Railway or at the Waterfront. Avon wishes to avoid Omar and Omar wants to find Avon. If they don’t meet, their relationship comes to nothing.
   (a) Describe this game in a normal form.
   (b) Find all of the Nash Equilibria for the game you wrote down in part a.

3. Find all the Nash Equilibria in the following Normal Form game. This might arise from the Fiscal Coordination game in from Lecture Notes 1, Practice Question 1.

<table>
<thead>
<tr>
<th></th>
<th>Country 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Deficit</td>
</tr>
<tr>
<td>Country 1</td>
<td>1,1</td>
</tr>
<tr>
<td></td>
<td>-2,-1</td>
</tr>
</tbody>
</table>

4. Imagine that you and a friend are considering meeting for lunch at noon today, but haven’t had time to plan in advance where to meet. Lunchtime is here, and now you must decide where to go without further communication. There are only two places where you can meet, the University Centre and Cadboro Commons. Both of you prefer to meet at either place than to not meet, but if you do meet you prefer the Cadboro Commons, while your friend prefers the University Centre.
   (a) Write the normal form for this simultaneous move game.
   (b) Identify the Nash equilibrium (or equilibria, if there are more than one.)

5. Nash Bargaining Game models a two person bargaining problem as a non-cooperative game. The players make simultaneous “demands” for a share of one dollar. (A share of a dollar is some number of cents between 1 and 100.) If those demands are mutually feasible (sum to $1 or less) they get their demand if not they get nothing.
   (a) Briefly explain why any two demands that sum to $1 will constitute a Nash Equilibrium.
   (b) Can you think of a Nash Equilibrium in which neither player gets any money?
   (c) If you were playing this game, do you expect that one particular Nash Equilibrium be a focal point solution? Explain.
Chapter 3: Sequential Games

The classic games in the previous lecture were all “one-shot” simultaneous move games, where players choose strategies without knowing the choice of their opponents, and the game ends after that one choice. There is no history of play to keep track of in these one shot games. In sequential games the players move in sequence. For all but the first player, there is a history of play that players can potentially use when choosing their moves. I say potentially, because it may not be true that players observe the previous moves of other players (or, perhaps, even moves they themselves made earlier in the game). Contingent strategies comprise moves for any possible way that actual play can unfold over time, so contingent strategies can be chosen simultaneously at the start of the game. This means that the solution methods we explored in lecture 2 work also for sequential games transcribed into the normal form.

However, the fact that play unfolds over time does introduce additional considerations, some of which can help us to select between the Nash Equilibria we find in the normal form. Since actually moving is different from announcing a plan, player who move first can credibly commit to playing a particular strategy. This may or may not prove advantageous. Later players may wish to make commitments, but be unable to do so. Nash Equilibrium is quite unrestrictive about credibility: player must play a best response, but equilibrium strategies can contain all manner of threats and promises of behavior “off the equilibrium path”. In this lecture we explore stronger equilibrium concepts that eliminate some unreasonable NE that rely implicitly on moves within the strategy following histories that do not occur in equilibrium.

It turns out that it may even be advantageous to a player to bind herself and limit her future choices. This is something that is not going to be true in standard single person choice theory, where some choices can be ignored, so more options are always better. Clever players can take strategic moves, moves that are alter subsequent play in their favour. This leads to the idea of an embedded game. We explore some simple ideas along these lines.

Finally, a special class of sequential games feature the same “stage game” is played multiple times; these are called repeated games. In principle, the stage game can be any well defined game, but most commonly it is a simultaneous game like those in the previous lecture. In the next lecture we take up this special case in some detail, and explore the implications of repeated play for some of the classic games from lecture 2.

Key terms: extensive form, subgame, proper subgame, refinement, subgame perfect equilibrium, backward induction, embedded game, forward induction, commitment device.
1. The trespass game

Brandon’s Beach Resort controls easy access to a public beach. Anna’s Adventure Tours sets up dive from the beach would find it convenient to use Brandon’s access. Brandon doesn’t approve, and has posted a no trespassing sign to deter people from using his access. If Anna ignores the sign, Brandon can sue for damages due to the trespass, but this requires going to court, and the case is costly to prove. If she did trespass and Brandon prosecutes, Anna will be convicted and face a small fine. The extensive form is:

Brandon moves only if Anna trespasses; otherwise the game ends after Anna’s move.

2. Subgames

A subgame is a part of a sequential game that could itself form a game: it must have a well defined set of players, strategies and payoffs. Subgames can be defined formally, but the basic idea is reasonably intuitive. Think of snipping off decision nodes from somewhere in a game tree. The decision node you snip out can become the initial node of a “new game,” provided, of course, that together with the following moves and nodes it forms a well-defined game tree. This new game is a subgame of the original game. Each subgame must begin at a well defined decision node and retain the entire portion of the original game that starts from that node. Careless snipping that cuts through a decision node, removing one or more actions, is not allowed; nor are cuts that split information sets, either at the new initial node, or later on in the game tree.

The trespass game has two subgames, indicated by the red boxes in the diagram below. After Anna moves, Brandon faces what amounts to a one-player game. The other subgame is the whole game. Calling the whole game a subgame may seem confusing, but clearly it doesn’t violate the rules above. (The whole game is what you get when you cut at the first decision node: i.e. don’t leave behind any nodes.) A subgame that is not the whole game is sometimes called a proper subgame. It is convenient to include the
whole game in the definition of subgame, as this makes it easier to describe properties that apply to all proper subgames and the whole game.

3. Nash Equilibria

Anna has two strategies in the trespass game: 1. Trespass (T) and 2. Don’t trespass (D). Brandon has two strategies: 1. Prosecute if Anna Trespasses (P|T) and Ignore if Anna Trespasses (I|T). Translate the game into a normal form and convince yourself that there are two Nash equilibria in this game:

1. \{T, I|T\}
2. \{D, P|T\}.

The following figures display the two Nash equilibria on the extensive form. You can verify that they are NE by checking each move in the player’s strategies to see if you can find a beneficial deviation. Remember that Nash equilibrium takes the other players move a fixed when checking for deviations, so only move one player’s arrows at a time.
4. Subgame Perfect Equilibrium

Many sequential games it has more than one Nash Equilibrium, and as usual we will look for refinements to eliminate some of these. The most widely used refinement in sequential games is called subgame perfection. Subgame perfection requires equilibrium strategies to be best responses to each other in every subgame (including the whole game). Clearly this is stronger than Nash equilibrium, which requires players to choose mutual best responses on the whole game, but not on every proper subgame. Choices in subgames “off the equilibrium path” do not affect the course of play, so changing moves there does not affect payoffs. These off the equilibrium path moves are therefore always best responses to the Nash equilibrium strategies. Subgame perfection demands strategies from a Nash Equilibrium, but also that player choose best responses in every subgame, whether on the equilibrium path or not. Because subgame perfection has these extra requirements, some Nash equilibria will not be subgame perfect. This is the sense in which subgame perfection is a “refinement” of Nash equilibrium.

You might think that the choices made at nodes that are not reached couldn’t possibly make any difference for anything. After all, this is the road not taken. Nash equilibrium implicitly assumes this very thing. However, if you think of best responses as best relative to beliefs, you’ll see that in this implicit assumption, Nash equilibrium allows for any beliefs about play that occurs off the equilibrium path. But just because a node isn’t reached, it doesn’t mean the move there doesn’t affect other players’ choices. It may be because of the move expected at some off equilibrium node that a previous player chooses in a way to prevent the node from being reached. In Nash equilibrium the beliefs must be “correct,” i.e. consistent with actions, but for unreached nodes this restriction amounts to nothing because no belief will be proven wrong. By fixing on what players do in equilibrium, and placing no restrictions on what they say they would do elsewhere in the game tree, Nash is overlooking something of potential significance.

You might even say this oversight violates the spirit of Nash equilibrium, at least by one account. Recall that one justification for using Nash equilibrium is that it is rational for players to use best responses, so rational players should expect mutual best responses and so a Nash equilibrium to be played. (In some cases, players might work the NE out and act accordingly.) In sequential games, Nash is too weak from this perspective, since all moves off the equilibrium path are best responses, “rational” players may select strategies that are themselves not best responses on those unreached subgames. Subgame perfection rectifies this shortcoming by requiring equilibrium beliefs about off equilibrium play to presume best responses, even on paths that are not reached during the course of play.
Subgame perfection rules out incredible (i.e. unbelievable) threats and promises. It requires that in equilibrium, players only anticipate choices that are optimal at the time of play. In contrast, Nash equilibrium allows players to “commit” to strategies that call for actions that are not best responses, as long as these actions are never actually chosen during the course of play. Under Nash, other players take these choices as given, and might then make choices that ensure the actions are never chosen. Hence, the “threat” implicit in the strategy is never carried out and therefore costless. Subgame perfection requires best responses on all subgames, even the ones never reached during the course of play. Players must only use strategies that threaten actions they would actually choose if they were called upon to play.

These issues are clear in the trespass game. When Anna chooses Don’t trespass, the payoff for Brandon is the same whether Brandon chooses to Prosecute conditional on Trespass, or to Ignore conditional on trespass. Since Anna doesn’t trespass, the payoff to Brandon for these two strategies cannot be different; his decision node is off the equilibrium path. (If you haven’t done so already, switch Brandon’s arrow in the diagram for this equilibrium above: you will see either way, Brandon gets a payoff of 0). Nash looks at payoffs, and only in one of the two equilibria does Brandon’s choice in subgame 1 directly affect equilibrium payoffs. But clearly Brandon’s choice in subgame 1 is relevant, even essential to the NE. When Brandon’s strategy threatens prosecution, Anna’s best response is Don’t trespass; when Brandon’s strategy doesn’t threaten prosecution, Anna’s best response is to Trespass. Anna’s belief about his intentions (encoded in his equilibrium strategy) determines her best choice. Nash equilibrium does not challenge these beliefs, but only requires them to be consistent with choices. In the equilibrium where Brandon’s choice is never actually taken (Anna doesn’t trespass), either of his moves is a best response to her action. Subgame perfection is stronger: it rules out the Nash equilibrium where Brandon Prosecutes conditional on trespass, because this is not a best response on subgame 1.

5. Backward Induction

In simple games, the easiest way to find subgame perfect equilibria is to use backward induction. Backward induction is a technique (or algorithm) for constructing subgame perfect equilibria. It is not equivalent to subgame perfection (which is a requirement, not an algorithm). In some games backward induction does not work, and other techniques are required to find subgame perfect equilibria. We will see examples of this in the next lecture. Backward induction involves finding Nash equilibria for each subgame, starting at the end of the game and working “backwards” through the game tree. Each time you move back a step, you fix beliefs about play in the future of the game according to the strategies chosen in previous steps, and find NE choices for the newly included actions. This process “induces” a subgame perfect equilibrium.
The base step for the induction occurs at the final decision nodes, right before payoffs are realized. These final subgames are well defined games with one player, essentially choice problems faced by the final player at the end of each branch. At this point, the choices of all other players have been made. By restricting choices here to a best choice for the final player, we guarantee that this choice is a best response to any set of strategies in the whole game: by construction the best choice at a final decision node is a best response to strategies that lead to it; when strategies in the game don’t lead to that node, then the choice there doesn’t affect payoffs, and so is also a best response the strategies in the whole game.

The next step of the induction moves backward from these final decision nodes to the subgames starting once step back, at decision nodes leading to the final choices. The choices in each of these “next to last” subgames each lead to one of the final subgames (of course, some actions might also lead directly to a terminal node). Player beliefs at each of these second level decision nodes are determined by the strategies computed in the base step for the final subgames. By choosing a best response to these credibly anticipated strategies, we find a Nash equilibrium on the subgames starting at the next to last nodes. (By construction, the strategies at the final nodes are best responses to any move that leads to the node.) Since players are choosing best responses conditional on arriving at each next-to-last decision node, again they are also playing a best response to any strategy on the whole game whether it leads to that node or not.

Working back a further step, the whole process repeats (this is the essence of “induction”). Players at the third-to-last nodes predict future play from the first two steps of the induction process (which are by construction best responses to any choice at the third to last nodes). To find a NE on these subgames, we find best responses to that expected future play. Again, these third to last choices will be best responses to any strategy in the whole game. And so on.

This process continues until we reach the very first decision node in the original game, where the whole game is the subgame. At that point, the best response will be a best response to the predicted future play, and all of those future choices will be best responses to each other and the first choice also. Thus backward induction will lead to a Nash Equilibrium on the whole game. But more than that, these strategies will be Nash on every subgame, and composed of choices that are optimal if they are ever called upon to be taken. We have eliminated “incredible threats”: everyone would carry out the plans in their strategy if they are ever in a situation to do so.

Subgame perfection is not an all purpose refinement. Sometimes the subgames themselves have multiple equilibria. In games with complex information structures this simple idea is also not generally enough to eliminate all unreasonable beliefs. Stronger restrictions are typically needed.
Subgame perfection in the trespass game

To see how this works in practice, look again at the two NE in the trespass game. Each involves a different choice by Brandon in subgame 1. These moves determine the equilibrium beliefs of Anna when she decides whether to trespass. Consider the equilibrium supported by the “threat” of prosecution. Is this threat credible? Should Anna believe it? Subgame perfection says that the answer is no, and we can eliminate it using backward induction. The base step looks at the final subgame, where Brandon gets to move. In subgame 1, the best response (i.e. the best choice) is to Ignore (I). Moving “back” to the next subgame, 2, which in this simple case is the whole game, we compute Anna’s best response using the beliefs about Brandon’s strategy derived in the base step. Since this has Brandon ignoring trespass, Anna’s best response must be best response to Brandon’s not prosecuting. This means she will trespass (T). Backward induction lead to the Nash equilibrium {T, I|T}, which is subgame perfect.

To summarise, in one of the two Nash equilibria in the trespass game, Brandon’s “policy” of prosecution serves his interest. Anna believes it, and it deters her from trespassing. It’s Nash, because Brandon never has to actually follow through on the threat, so it is costless (beyond the price of a “trespassers will be prosecuted” sign). But this equilibrium is not “reasonable”: payoffs are common knowledge, and Anna should realize this threat is not credible. Since Brandon is worse off following through on the threat, Anna should see through it, disregard the sign and trespass. This is exactly what happens in the game’s unique subgame perfect equilibrium.

6. The centipede game

The centipede game illustrates that subgame perfection’s requirement that players choose mutual best responses on every subgame rules out promises as well as threats, when honouring these promises is not in the interest of players. The centipede game features a lengthy series of alternating moves. At each move, the player whose turn it is can stop the game for an exit payoff, or pass to the next player. (The example below has four decision nodes; the game’s name suggests a hundred would be more appropriate. The diagram is thought by some to look like a centipede with the exit actions as legs and the pass actions as links between body segments.)
In this game, a conflict emerges between individual incentives and collective welfare. The total value grows every time a player passes, but individually each player can do slightly better by ending the game rather than pass it to the next player, if that next player is expected to end the game. Backward induction is very powerful: the final player benefits by ending the game rather than passing to the final terminal node at where total value is largest. Predicting this, the next to last player will choose to end the game. This process repeats and the equilibrium unwinds to the point where the choice every node is predicted to end the game, including the very first. Convince yourself by working backwards through the extensive form above. Subgame perfection rules out any gains from waiting for the total value to grow. Experimental evidence suggests that this bleak prediction is not always obtained.

7. Strategic Moves

Many of the games we play are fixed and beyond our control. However, sometimes you may find yourself in a game that you cannot win, and you should ask yourself if you can change the game. This might, for example, involve dreaming up new strategies or involving new players. In essence we take a game with an outcome we don’t like, and embed it in a larger game with an equilibrium we prefer. (Of course, the fact that you can add strategies implies that you already were in fact playing a larger game, where adding strategies was itself a possible action: in a deep sense it might be that we can never really change the game.) These game-changing kinds of actions that improve the prospects of a player are sometimes called “strategic moves”, which is appropriate enough, but should not be confused with our concept of a strategy.

We can examine this idea by adding another move to the centipede game. To keep it simple we think of a horse rather than a centipede, and let each player move only once. The trust issue highlighted in the centipede game becomes all about player 2. Even in the reduced game, backward induction again leads the prediction that the game ends (inefficiently) on the first move.

Consider how the game might change if we add a move prior to that of player 1, in which player 2 can choose to post a bond worth B units of payoff utility. Assume the bond is placed with some trustworthy third party, who agrees to return the bond to player 2 if and only if the game reaches the final node. (Why this third party is
trustworthy while player 2 is not shall be left unanswered; many lawyers earn a living in part from their reputation for doing just this sort of thing, so it is not unreasonable to assume these situations could exist.) If the bond amount B is large enough, it is no longer in player 2’s interest to exit at her turn. Exit grabs the exit payoff, but forfeits the bond. This gives player 1 the confidence necessary to pass the game to player 2. Knowing this, it is then reasonable for player 2 to post the bond in the first place. Now the subgame perfect equilibrium allows the efficient outcome to be reached, and the bond is always returned.

The bond is a specific example of a commitment device. Player 2 adding the bond posting stage is a strategic move. It directly affects payoffs and changes the equilibrium.

Appendix: Forward Induction

The following example from McCain(2004) demonstrates how thinking about the best responses that players made before your move can refine the set of equilibria. In backward induction players think ahead and work backwards from the end to the start of the game. In this example, players think back to earlier moves and work forward to the present move of the game; hence it is called forward induction. Forward induction is a more general term covering various refinements based on “reasonable” restrictions on beliefs about the unknown moves earlier in the game. This more general exploration rapidly gets complex, requiring technical sophistication beyond the scope of this course. In this clever example, the beliefs in question are the very same ones we have already seen: uncertainty about the simultaneous choice of another player. These notes are included as an aside. I do not intend to spend class time on this subject, and will not cover it on an exam.

A1. What do we do this weekend?

Adam is thinking about visiting Brenda for the weekend. If doesn’t go, he can stay home and work in his garden. If he does go, they have a choice of going hiking or sailing. Adam needs to travel to meet Brenda, and unfortunately, before he leaves, Adam and Brenda are cannot make a specific plan to either hike or sail. At most, Adam can send a simple message saying whether he coming or not.

First consider the payoffs from the weekend activities. In a moment we’ll embed these in the larger game including Adam’s choice of whether to visit Brenda or tend his garden.
These payoffs indicate that if Adam visits, both players prefer to do something together rather than not, but they disagree on which activity is best. When Adam comes prepared to sail, but Brenda has prepared to hike (or vice versa) they get the lowest payoff. As you see, this is simply the coordination game, BoS. There are two Nash Equilibria, each of which is preferred by one of the players.

Next we embed this in the whole game that includes Adam’s choice whether or not to visit. Using an information set to represent the simultaneous choice in the BoS subgame, the extensive form is

Notice in this game backward induction does not reduce the prediction to a single NE. Because we have multiple equilibria in the activity choice subgame, there is no unambiguous outcome in the subgame that can be used to construct Adam’s best response at his first node. Both NE are reasonable, and depending on which he expects it may be better for him to visit Brenda (for a sail) or stay home in the garden.

A2. Forward induction

Although it may seem like we’ve got nothing from considering the larger game, as you will see, it does open up some interesting possibilities. The idea we will consider is called forward induction. Like backward induction, forward induction involves thinking about what is “reasonable” to believe about other players. But instead of providing a reasonable prediction for future behavior, forward induction aims to provide a reasonable interpretation of past behavior.

This can all get rather confusing. Backward induction looks ahead in time while forward induction looks back at history. But remember the algorithmic nature of these ideas. You start at one end of the game and work toward the other end, at each step imposing some plausible reasoning about choices, which are then used to restrict beliefs at the next step. Backward induction works from the final move to the first move, backward through the game tree, restricting beliefs about the future. Forward induction works from the first move to the last move, forward through the game tree, “restricting” beliefs about the past.

Since past behavior is in the past and observed, forward induction does not in any way affect beliefs about these moves. What else is there to know? Forward induction uses past choices to assess the beliefs (expectations or knowledge, or in some cases, preferences) of the players who made those choices. Since the observed moves are best responses, they contain information about the beliefs of the players who make them. To be precise about these ideas required a better model of beliefs than
we have so far developed in this course. And even with the most careful explication, “reasonableness of beliefs” is a slippery concept. Nevertheless, the example above can give a sense of how this works.

To use forward induction, we start with Adam’s first choice. As we saw, what he does here depends on his beliefs about what will happen in the activity subgame. As a result what his choice reveals information about what his beliefs about what will happen in that subgame. Then in the coordination game, Brenda can use this choice to infer what she should expect from Adam and coordinate her actions. What should she assume about his expectations if she learns that Adam intends to visit? Should she ever expect Adam to visit, prepared to go for a hike?

To put this in the narrative form, Brenda’s forward induction argument for beliefs about Adam’s choices in the activity choice subgame might be something like the following:

I know that Adam prefers it when he Gardens alone to when he Hikes with me (4>2). Thus the only reason he would have left the message that he is coming is that he expects that we will go Sail, not Hike. Therefore he must be preparing to Sail, and hence so should I.

This works for Adam also, since he can work out Brenda’s forward induction argument for himself. He can reason that the message he is coming effectively signals that he is planning to Sail. So he can also make a choice in the subgame with confidence in his belief about Brenda’s choice.

This eliminates one NE from the subgame, and allows for players to get the payoff from (Sail, Sail).

Practice Questions:

1. An old game with two players starts with 21 matches placed in a row on a bar. The players alternate removing matches. At each turn, a player can remove either one or two matches. The person who takes the last match loses. Generally, the person offering the game will let the other player go first. Let’s call the two players Angel and Billy. An extensive form for a simplified version of the game that starts with only 5 matches on the bar might look as follows:
(a) How many strategies does Angel have? How many does Billy have?

(b) Find a subgame perfect equilibrium.

(c) Now consider the 21 match version. Is the five match game a proper subgame of the 21 match version?

(d) In the 21 match game, how many matches should Angel take on the first move? Assuming she does this, and at some later move Billy takes one match, how many should Angel take on her next move (assuming Billy didn’t just take the final match)?

2. The Oregon Question. In her book *Deterrence through Strength*, Rebecca Berens Matzke describes the role of the British Navy in the defence of Canada during the middle part of the 19th Century:

   The Oregon Territory would be a key part of the broader Pacific trade for both Britain and the United States. An earlier treaty had given the two countries joint rights of occupation in Oregon, and previous agreements had not decided on a border. When US Democrats tied the annexation of Texas to that of Oregon in the 1844 presidential campaign, the issue moved to the forefront of Anglo-American relations. This political manoeuvre stirred up expansionist enthusiasm in the United States and thereby, as [British Prime Minister] Peel observed, made “compromise and concession (difficult enough before considering what stands on record of past negotiations) ten times more difficult now.”

   Peel and his foreign secretary, Lord Aberdeen, ... had no patience for the blustering of US presidential candidate James K. Polk and his supporters. Peel favoured arbitration, not concessions to the United States, and he thought the best answer to American belligerence was to sent the Pacific squadron flagship *Collingwood* ...to make a “friendly visit ...to the mouth of the Columbia [River]”

   ...When the British minister in Washington misunderstood his instructions and rejected ad American proposal in 1845, a furious [now President] Polk renewed calls for a US border above the forty-ninth parallel. Peel commented to a friend in January 1946, “We shall not reciprocate blustering with Polk but shall quietly make and increase in the Navel and Military and Ordnance [budgets]”...The great disparity between the British capability of attack and the American capability to
defend gave Polk the political excuse he needed to break with the confrontational posture of his party, and the US agreed to Aberdeen’s Oregon boundary offer in 1846 [the present BC/Washington State boundary]¹³.

(a) Formulate the Oregon Border dispute as characterised by Matzke in an extensive form game.

(b) What must be true about the increase in British armaments so that this is a case of credible deterrent?

(c) Indicate the subgame perfect equilibrium for your game.

3. Consider the following extensive form.

(a) How many strategies does player 1 have?

(b) Find the subgame perfect equilibrium. Is it unique?

(c) What if anything is wrong with the following reasoning? The combination of strategies (D and U|UA and D|DA) for player 1 and (A|U and D|D) is a Nash Equilibrium. Since the equilibrium calls for player 1 to play D, switching the move following UA makes no difference to payoffs, so that is a best response. Meanwhile, since the equilibrium move has 1 play up following UA, switching the move at the first node from D to U leads to a payoff of 2 instead of 3, so this deviation lowers payoffs. Finally, since 1’s node following

DA is also not reached, again switching moves doesn’t improve payoffs. Hence this is a Nash Equilibrium.
(d) How, if at all, would your answer to part c change if the payoff following the path DD were (5,9) instead of (3,9)?

4. A common way to divide objects among children is by assigning one as the splitter and the other as the chooser. The splitter partitions the object, and then the chooser gets to select their preferred piece, leaving the other for the splitter. Think of the object as a cake.

(a) Write this game as an extensive form, assuming that the cake only be divided into shares of (0,1), (0.25,0.75), (0.5,0.5).

(b) Find all of the Nash Equilibria for the game you wrote down. Which if any are subgame perfect?

(c) Does either player have an advantage in the subgame perfect equilibria you found in part (b)? If so who, if not why not?

(d) Assume now that players disagree the value of one of the quarters of the cake (it has more icing), but agree on the value of the other three quarters. One on the children, who loves icing, values the quarter with more icing at three times the worth of each of the three other quarters. The other child (who scrapes off all the icing anyway) values it exactly as much as each of the three other quarters. Now is it better to be the divider or the chooser?
Chapter 4: Repeated Games

Repeated games are a special class of sequential games in which the same stage game is played repeatedly. Repeated games are sometimes called “supergames”. Because the same basic game is played over and over, repeated games retain a fairly simple structure even when repeated indefinitely. However, because strategies can be history dependent, repeated games can be used to study very complex strategies. These games are well suited, for example, to study punishments and rewards for cooperation. Repeated games can also be used to think about evolution and learning, as we will do later in the course.

Key words: stage game, Folk Theorem, tit-for-tat, grim trigger, time preference, discount rate

1. Repeated Prisoners’ Dilemma

One of the obvious criticisms of the prisoners’ dilemma (and social dilemmas more generally) is that a one shot game doesn’t allow for the cooperation “naturally arising” in a long-term relationship. Recall the story:

- Two suspects, Albert and Brian, have been arrested
- They are put in separate holding rooms by the arresting detectives
- Each is offered a deal if he provides evidence against the other
- Without this testimony, conviction on the main charge is impossible, but some lesser offence can be proven

The extensive form for this simultaneous move game is

```
Albert
/  \            (1,1)
|   |
Defect  Cooperate

Brian
/  \            (3,0)
|   |
Defect  Cooperate

Brian
/  \            (0,3)
|   |
Defect  Cooperate

(2,2)
```

Notice that the strategies have been relabelled “Cooperate” and “Defect” to accord with the more general interpretation of social dilemmas. (When we first met this game,
Cooperating was labelled “don’t confess”, and Defecting was labelled “confess”). In the one shot game, both players have dominant strategies to Defect.

If we repeat the game, so it is played twice\(^\text{14}\), it has five subgames: the whole game, and four proper subgames, one starting at each of the possible histories of play from the first round. The extensive form is:

With just this one repetition, the number of strategies grows dramatically. Each player now has \textbf{thirty two strategies} to choose among, one pair of which is illustrated in the extensive form above. The number of strategies more than doubles with the extra round of play because with repetition players can use contingent strategies (of the \textit{if…then} variety). These strategies, at least in principle, offer the possibility to punish non-cooperative behaviour.

2. Tit-for-Tat

Players need not pay attention to history, and can always play simple history independent strategies such as “always defect”. Strategies that do condition on the history of play may also remain quite simple. This is true even when a game is repeated many times.

For example, the diagram above illustrates the contingent strategy tit-for-tat

\(^\text{14}\) Somewhat confusingly, I will call this the twice repeated game.
Round 1: Cooperate.
Round 2: Cooperate if the other player Cooperated in the round 1; Defect if the other player Defected in round 1.

Tit-for-tat is “generous,” in that it starts by cooperating. It remains so if it the opponent is similarly generous, but tit-for-tat players aren’t patsies and will punish defection.

Tit-for-tat generalises to games with more repetitions, remaining very simple. In the first round, it starts with cooperating. After that, it chooses moves that depend on (and identical to) the previous move by the other player. It chooses cooperate following cooperation and defect following defection. Even as histories lengthen with further repetition tit-for-tat looks back one round to the other player’s most recent move. One way to think of “simple” is the restriction on how much the strategy needs to “remember”: tit-for-tat only needs to remember the most recent play of the other player. It ignores the tit-for-tat player’s own history of play, and doesn’t depend on history of outcomes.

3. Subgame perfect equilibrium

Now consider the subgame perfect equilibrium of the twice-repeated prisoner’s dilemma. In the final period whatever happened in round one is in the past, and nothing players do in the second round can affect the payoffs earned in the first. Because the game is coming to an end, there is no future to look ahead to. Each prisoner in the last round, therefore, has the same incentives as he does in the one-shot game: the (Dominant Strategy) NE is (Defect, Defect). It is easy to verify that the arrows in the subgames following histories of cooperation that indicate further cooperation are not part of any Nash equilibrium in those subgames. The “promise” implicit in tit-for-tat to cooperate in the second round is not credible. If the promise of tit-for-tat were believed in the first period (which it won’t be in a SPE), then cooperation in period 1 would be optimal. But, accounting for best responses in period 2, it is easy to verify that best response for both players in the first round is also Defect. The game’s unique subgame perfect equilibrium has both players Defecting in each round.

In general, the incentives to cooperate in games that are repeated a known, finite number of times break down in the final period of play. In the last period there is no future, so promises became ineffectual. With the last period play determined by its own short term considerations, there is then no role for promises in the second to last period: with future actions known, reputation, threats and promises have no traction. Therefore, in the next to last round, players again choose a NE of the stage game, just as they do in the final round. The future exists, but since play is determined there, it cannot be used to reward cooperation or punish defection. This process repeats itself at
the third to last period, and so on. The whole structure of threats and promised “unravels” back to the very start of play. Because of the last period incentives, and this unravelling, the whole set of complex contingent strategies do not add any credible alternatives to the players’ basic set of strategies in the stage game.

This unravelling would not occur if there were no last period.

4. Indefinitely repeated games

How can we model a game that never ends? A technical problem immediately emerges. If the game goes on forever, over time the payoffs add up. Strategies affect how fast payoffs accumulate, but if payoffs are positive, many combinations of strategies eventually lead to an infinite payoff. If this happens, all such strategies are equally good. To avoid this, we need some reason why payoffs from various strategies add up to a finite numbers, which then can be compared.\(^\text{15}\)

To get an endless series of payoffs to sum to a finite number, certain conditions must hold. Intuitively, most of them must be very, very small, essentially zero. It makes sense to assume that the tiny payoffs are those way off in the future. Since payoffs are about preferences over outcomes, not the outcomes themselves, what we need is for players to not care very much about outcomes in the distant future. And in many contexts, it make sense to assume that the further off in the future the outcome is, the less a player cares about it.

There are two essentially equivalent ways to interpret the assumption that the future matters less than the present.

Discounting

Discounting directly assumes that players care less about time periods in the distant future. The game goes on forever, but it is one thing to say that there is always a tomorrow, and quite another to say that players are willing to forgo short-term benefits for payoff in the distant future. Preferring current outcomes and those in the near future to those further off in time is called “discounting” the future. The more you discount, the less you care about tomorrow, and so on, with the weight on a payoff diminishing as the period the payoff is earned gets further into the future.

\(^\text{15}\) There are other ways to solve this problem that we don’t explore. For example, players could care about the long run average of their payoffs.
Discounting makes particular sense with financial matters, where interest rates indicate the time value of money. But players may discount future utility also. The discount rate refers to the rate expressing this preference for present payoffs. If players discount future payoffs sufficiently, the game can literally go on forever, yet different strategies have different “present discounted values”. Therefore, players can choose between strategies, and best responses are well defined.

**Random end time**

Another possibility is that the game does end sometime, but players don’t know exactly when the end will come. The simplest case is when the game ends with some fixed probability in each period, say based on the toss of a coin. Every period might be the last, but it is always uncertain. Thus players face a lottery, and we assume they choose strategies to maximize expected utility.

If the chance of ending is very high, almost certain, then the indefinitely repeated game is very similar to a one shot game. If the chance of ending is low, it is similar to one that doesn’t ever end. As the chance of ending varies between 0 and 1, the degree to which players “care” about future payoffs also varies. Strategies that lead to payoffs in the distant future fall in expected utility relative to those that yield the same payoff in earlier stages. Not because the payoffs themselves are discounted, but rather because these more distant periods may never arrive. Assuming a random end time is perhaps more realistic, but it is mathematically equivalent to discounting.

**Strategies with an infinite number of moves**

Strategies in indefinitely repeated must specify a choice for each possible round of play. Even games that end each period with some probability (less than 1) can always keep going, so the strategies cannot assume they end. Not surprisingly, these strategies can be very complex, as the take into account more information about past play. But as with finite repeated games, simple strategies still exist. The simplest strategies involve moves that are identical in each period, or change mechanically, but the more interesting are contingent on the history of play. These can depend on histories of various lengths. For example, tit-for-tat is a simple strategy that depends on a one period history, and only looks at the choice made by the opposing player. You could imagine others that started cooperating and then punished non-cooperative behaviour.
only if it lasted for two periods (tit for two tats). Still others take into account the player's own previous choice, but depending on the outcome in the previous round.

5. The Folk Theorem

When the future is unimportant, either because the game is highly likely to end or because players discount heavily, a repeated game becomes very like a one shot game. (If it ends with certainty after one period, it is the one shot game.) However, if the future is sufficiently “important”, then large enough threats and promises can support cooperative equilibria. This is, very loosely speaking, the “Folk Theorem”. The size of the required threat (or promise) is a function of the payoffs in the game. The folk theorem requires threats to be credible, and so it must be a subgame perfect equilibrium to carry them out. The larger the credible penalties that players can impose on other players who “cheat” the less must the future matter in order to enforce cooperation.

6. Indefinitely repeated prisoners’ dilemma

Assume that the game may end each period with some probability \( 1 - p \), a probability that does not change over time. It is as if at the end of each period, someone tosses a coin to decide whether another round is played. For now, we only assume that \( p \) is some probability, not necessarily 0.5 as it would be for a coin toss. A more general model might allow \( p \) to change over time, perhaps decreasing, as the players get old. But notice, even very old players might expect to play again with some small probability (i.e. that the game does not end with probability 1).

In this potentially infinite game, strategies must indicate a choice for each of the potentially infinite number of periods. As noted above, this can allow for extraordinarily complex strategies. It turns out that very generally we can ignore most of the complex strategies, because game theorists have proven that whatever can be achieved with a complex strategy can also be achieved with a relatively simple strategy that uses only one period of history.

The first possibility to consider is that the players choose to play the dominant strategies from the stage game. This is a very simple strategy, independent of the history of play: Always Defect. The expected utility is

\[
EU(\text{Always Defect}) = 1 + p \cdot 1 + p^2 \cdot 1 + p^3 \cdot 1 + \cdots
\]
This is an “expected utility.” It is found by multiplying the payoff from each period by the probability the period occurs, and then summing across all periods. It is as if we were taking the “expectation” of the payoffs utilities. This expression can be written more simply as:

$$EU(\text{Always Defect}) = \frac{1}{1-p}$$

From this you can see that as the probability of playing again goes to 1, so the game is almost surely played forever, the Expected Utility becomes very large. When the game ends for sure, the payoff is that of the one shot game.

NOTE: Knowing how to manipulate these sorts of sums is essential if you want to solve games that have a potentially unlimited number of periods of play. However, for the purposes of this course, concentrate on the ideas rather than the arithmetic; I’m including this largely because if this algebra makes sense to you, the ideas are typically much easier to understand when presented in this manner. Some of the exercises below ask you to do this, but I will not be emphasizing these computations on the exams.

**Nash Equilibrium**

Given the infinite number of other strategies available to players, our usual routine of considering each combination in turn is impossible, as is checking every conceivable deviation to see if a player is better off. Fortunately, we can rely on some general theorems that tell us we can check for Nash Equilibrium by verifying only that “one period deviations” do not increase payoffs. The result implies that if any deviation improves the payoff for a player, then some one-period deviation improves the payoff. This greatly simplifies the task of checking for equilibria.

Consider a one period deviation by Albert. Given that Brian continues to play the dominant strategy from the stage game, the Expected Utility of switching from Defect to Cooperate for one period is

$$EU_{dev} = 0 + p \cdot 1 + p^2 \cdot 1 + \cdots$$

Albert’s utility falls in the period of his cooperation (since Brian continues to Defect). After that things revert to the (Defect, Defect) dominant strategy equilibrium.

The Expected Utility for the one- period deviation is strictly less than the Expected Utility from staying with the strategy Always Defect. Therefore Always Defect is a Nash Equilibrium. As claimed above, this is all we need to check. In this case it is quite easy for you to yourself that longer deviations only make things worse for a player.

**Subgame perfection**
Clearly, backward induction will not work for this game, since there is no last period in which to start the induction. But we can still look for subgame perfect equilibria. This is because every subgame is identical to the whole game. This implies that the NE we have found is subgame perfect. Notice that this is a direct argument. In effect, we are checking that Always Defect forms a NE on every subgame. Even though there are infinitely many of these subgames, the task is simple because they are all identical.

7. Tit-for-Tat

Consider now the possibility that both players choose tit-for-tat. They start with Cooperate and then choose Cooperate if their opponent chose Cooperate in the previous period and choose Defect if their opponent chose Defect in the previous period.

When both players use tit-for-tat they always cooperate. The expected utility for tit-for-tat is

$$EU(TFT) = 2 + p \cdot 2 + p^2 \cdot 2 + \cdots$$

Is it a Nash Equilibrium for both players to use tit-for-tat? Again consider a “one period deviation” by Albert. Given that Brian continues to play tit-for-tat, the best Albert can hope for by Defecting for period 1 is\(^{19}\)

$$EU_{dev} = 3 + p \cdot 0 + p^2 \cdot 2 + p^3 \cdot 2 + \cdots$$

For tit-for-tat to be a Nash Equilibrium it must be that the payoff from staying with tit-for-tat is larger than the payoff from the one period deviation.

That is, for tit-for-tat to be a best response to itself, it must be that

$$EU(TFT) \geq EU_{dev}$$

$$2 + p \cdot 2 + p^2 \cdot 2 + \cdots \geq 3 + p \cdot 0 + p^2 \cdot 2 + \cdots$$

This looks very strange, no doubt\(^{20}\). But notice that beyond the first two terms, each side is identical (because both players, by assumption, revert to playing tit-for-tat).

\(^{19}\) This assumes that Albert forgives Brian’s punishment for defection, and lets cooperation be restored. Alternatively, this initial defection can lead to rounds of tit-for-tat retaliation. This outcome is no better, as you can verify.

\(^{20}\) You could instead work out the values for these summations.
Therefore we can just cancel out those terms (even though there are infinitely many of them, this works).

\[ 2 + p \cdot 2 \geq 3 \]
\[ p \geq 1/2 \]

As long as \( p \) is large enough then tit-for-tat is a Nash equilibrium. Here “large enough” means that the game is as likely as not to be played at least once more. The “promise” to cooperate backed up by a threat of a one period punishment in the case of cheating does form Nash equilibrium. With a little more algebra, you can convince yourself that longer bouts of cheating are even worse for a player than “one-shot” deviations like we just considered. If a one period deviation doesn’t lead to higher payoffs, longer deviations will not.

Finally, since every subgame is identical to the whole game, Nash Equilibria are again subgame perfect.

8. Other equilibrium strategies

Tit-for-tat is a so-called trigger strategy: behaviour changes discretely after a specific “triggering” event. Trigger strategies have the advantage of being simple.

Tit-for-tat is not the only trigger strategy, and other trigger strategies can also form equilibria in the repeated prisoner’s dilemma. For example, cooperation can also be supported by so-called “grim trigger.” This has players start by cooperating and then, if the other player defects, reverts to the dominant strategy from the stage game forever more. No forgiving here.

The key comparison is between cooperating and facing a lifetime of non-cooperation should you defect once. It is straightforward to show that this stronger punishment means that players can cooperate even when the future is not sufficiently certain to enforce cooperation under the milder tit-for-tat strategy.

Notice that this is a dangerous strategy: any error leads to the bad outcome. In contrast, tit-for-tat is more forgiving.

9. Stag Hunt and the Prisoner’s Dilemma
Political theorists have suggested that social dilemmas are avoided through the use of “Social Contracts,” implicit agreements enforced through repeated interactions. Thomas Hobbes argues in the book *Leviathan* that “fooles” who follow their short term interest to not cooperate in society err in ignoring the future. Society can punish defectors by never again cooperating with them. This is very much the logic of the Grim Trigger strategy.

As mentioned in Lecture 2, the Stag Hunt game is considered by some game theorists as a better model than the Prisoner’s Dilemma of the difficulty of coordinating on efficient strategies. In the Stag Hunt, the efficient outcome can be obtained as an equilibrium in a one-shot game. We saw that the “payoff dominant” equilibrium can be the outcome if players have sufficient “trust” in each other. It’s always easier to trust someone if you have reasons to believe it is in their interest to maintain your trust. And as we have seen, social efficiency can be achieved in the repeated Prisoner’s Dilemma, if players use punishment strategies. You might wonder if the Stag Hunt and the repeated Prisoner’s Dilemma can be linked more formally.

The political scientist Brian Skyrms thinks Stag Hunt is the better model of most social dilemmas, and he has devised version of the Repeated Prisoner’s Dilemma that reduces it to the Stag Hunt. Consider the repeated Prisoner’s Dilemma example above. Assume that the probability of each additional round is 2/3. Finally, restrict your attention to the two simple strategies, (Grim) Trigger and Foole, where Foole is the fixed strategy Always Defect. Let $EU(T, F)$ be the payoff from using Trigger against and opponent who plays Foole.

$$EU(T, F) = 0 + \frac{2}{3} \cdot 1 + \frac{4}{9} \cdot 1 + \cdots = 2$$

Similarly,

$$EU(T, T) = 2 + \frac{2}{3} \cdot 2 + \frac{4}{9} \cdot 2 + \cdots = 6$$

$$EU(F, T) = 3 + \frac{2}{3} \cdot 1 + \frac{4}{9} \cdot 1 + \cdots = 5$$

$$EU(F, F) = 1 + \frac{2}{3} \cdot 1 + \frac{4}{9} \cdot 1 + \cdots = 3$$

Assuming that these two simple strategies are the only choices for players, this can be organized into a strategic form game matrix:

<table>
<thead>
<tr>
<th></th>
<th>Trigger</th>
<th>Foole</th>
</tr>
</thead>
<tbody>
<tr>
<td>Foole</td>
<td>3,3</td>
<td>5,2</td>
</tr>
<tr>
<td>Trigger</td>
<td>2,5</td>
<td>6,6</td>
</tr>
</tbody>
</table>
This is a version of the Stag Hunt Game. For an exact analogy to the version we saw in Lecture 2, Trigger would have to be indifferent between playing Foole or another Trigger. That is not essential for the point here: both (Foole, Foole) and (Trigger, Trigger) are Nash equilibria. (Trigger, Trigger) is Pareto efficient and better for both players than (Foole, Foole). The trust between “Triggers” is in this case explicitly modeled as coming from the punishment inflicted by this strategy on players who defect.

Practice Questions:

1. Consider a Chain Store that operates in 100 locations. At each location the store faces one potential entrant. If the entrant chooses to enter, the chain store can either fight a costly price war or accommodate the entrant and share the market. If in any location the entrant does not enter, the chain store can continue to enjoy its monopoly profits in that market. Everyone knows that the monopoly profits are the best the chain store can earn, but that sharing the market is also profitable for each firm. Everyone also knows that a price war is the worst outcome, leading both firms to lose money.

Assume that these entry decisions are taken in sequence across the 100 locations.

(a) Model the stage game for one market as an extensive form. What is the subgame perfect equilibrium for that stage game?
(b) A consumer lobby group argues that even though a single store would prefer not to fight and would share the market, the chain store has different incentives. It would be happy to lose money in a small number of markets to build a reputation for fighting entrants. Does this claim make sense in your model?

2. Using the payoffs from the following version of the Prisoner’s Dilemma,

<table>
<thead>
<tr>
<th></th>
<th>Confess</th>
<th>Don’t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>-10, -10</td>
<td>0, -20</td>
</tr>
<tr>
<td>Don’t</td>
<td>-20, 0</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>

(a) show that Tit-For-Tat can sustain cooperation (Don’t, Don’t) as a subgame perfect equilibrium if the probability of repetition is greater than 1/19.
(b) Since these two games have the same ordinal preferences, how do your account for the difference from the example in the notes above?
3. What would happen to the relationship between the Stag Hunt and the repeated Prisoner’s Dilemma derived in section 9 if the probability of repetition were lower? Say, for example, 1/3? (You might answer this formally, but could argue instead from the results worked out in the sections above.)
Chapter 5: Co-operative Games

When we studied the Prisoner’s Dilemma we referred to the efficient outcome of cooperation as the *cooperative solution*. Since the Prisoner’s Dilemma is non-cooperative, and each player has the unilateral incentive to deviate to their dominant strategy, the cooperative outcome is not a Nash Equilibrium. Its demonstration of the tension between individual and collective benefits is the principal value of the prisoner’s dilemma game. Nevertheless, as we saw, when the prisoner’s dilemma is modified to allow players to make a commitment -- for example, by allowing indefinite repetition -- cooperation can be sustained.

In this lecture, we address the issue of cooperation from a different direction. Cooperative game theory *takes it for granted* that some enforcement mechanism exists to allow players to achieve the cooperative solution. It assumes players can commit to actions that are in their interest. Attention shifts from the maintenance of cooperation to the question of how agreements to cooperate are *structured*. The enforcement mechanism is not specified in the description of the game: agreements are simply assumed to be upheld. In cooperative game theory the focus is on which cooperative outcome will be agreed to by players, not how players make choices to sustain the agreement.

Keywords: Cooperative solutions, bargaining problem, coalition, worth, the core, Shapely value.

1. Cooperative Game Solutions

Since the goal is to understand outcomes rather than choices, *cooperative solutions refer to outcomes* (or sets of outcomes). Many simplifications help make these games tractable. It is common to assume “free disposal”. This means players can always dispose of some of their payoff, so can never receive too large a share: if they get more than they want, the freely dispose of the remainder. An important division is between games in which payoffs can be divided arbitrarily between players and those which they cannot. The former are called *transferable utility games* and the latter *non-transferable utility games*.

Intuitively, transferable utility games have something like works like money: it is finely divisible with small units, each of which is equally valued by the players. This currency can be used to re-divide the value of any outcome among players. In non-transferable utility games there may be some outcomes that advantage one player more than it disadvantages another, yet the “winners” have no means of bribing the “losers” into agreement. With transferable utility, the solution can characterised by the total value of the outcome. This simplifies the description of solutions and many of the key definitions. **We will focus on transferable utility games.**
2. Two player cooperative games: bargaining

The back and forth of negotiation can be modeled as a non-cooperative game, at least in principle, and in practice for some simple bargaining scenarios. The usual analogy is to dividing a cake. The cooperative game theory approach does not model the strategic back and forth, but asks instead, if negotiating partners can make binding agreements, what agreement will they make? How will the cake be divided?

Divide the dollar

A simple case of bargaining has two people deciding how to divide a dollar. If they can propose a feasible division, i.e. one where the total of the shares is less than or equal to $1, then their proposal is implemented. Otherwise they each get nothing.

For the sake of time, the class lectures will not cover the material in the rest of this section in detail. I include it here as bargaining is a familiar setting, and this discussion illustrates the methods of cooperative game theory.

Nash Bargaining

John Nash was an early contributor to the bargaining literature and wrote about this simple situation. He set out its essential elements, defining what has come to be known as a Nash Bargaining Problem. (It is important that you not confuse this work, and Nash’s fundamental contributions to cooperative game theory, with his fundamental contribution to non-cooperative game theory, the concept of Nash Equilibrium. These are unrelated except by the connection to John Nash, creating confusion to generations of students.)

A Bargaining Problem is defined by

- a description of the set of feasible outcomes;
- a description of the preferences over these outcome of the parties to the bargain. Preferences must differ over some of the possible outcomes (otherwise, harmonious agreement is assured, and there is no need to “bargain”); and
- a “disagreement point” the outcome that occurs when no agreement be reached.

When there is money (or something similar) to transfer utility, the feasible set is simple. In divide the dollar the feasible set includes all the possible ways to divide the dollar between the two players. To ensure utility can be transferred, we must also assume that for each player the value of an additional penny is the same for the first and the last penny. Players disagree, since each prefers a larger share of the dollar. The disagreement point is zero cents for each player.
Notice that we could proceed by defining a set of strategies available to the bargainers and then construct and solve a non-cooperative bargaining game. The outcome will depend on the details of the game, of course, and exactly which strategies we include. For example, we could assume that each player writes down a “demand” and then a referee opens these. If the demands sum to a dollar or less, the players get what they ask for; otherwise they get nothing. Or instead we might assume that players make alternating offers: a coin toss determines who goes first and then players take turns suggesting feasible allocations. Once the player receiving the offer agrees to it, the game ends and the money is divided according to that agreement. We need to say what happens while the parties bargain and in the event they cannot come to any agreement. For example, we might assume that during bargaining or in the case of breakdown, they get nothing. Then we could look for a Nash Equilibrium to predict what players choose to do.

Cooperative game theory takes an entirely different approach. It instead asks directly about the kinds of outcomes we might expect to see. Perhaps the simplest requirement is that the solution should be Pareto Efficient. If a proposed solution isn’t efficient then one of the bargainers can make a new proposal that makes one or both parties better off: unless this is somehow prevented, this new outcome should be adopted in preference to the original proposition.\footnote{This insight is sometimes referred to as the Coase Theorem, although it is perhaps more accurately a definition than a theorem. The Coase Theorem’s main value is in focussing attention on the characteristics of a situation that prevent bargaining from reaching an efficient settlement. These impediments are called “transaction costs” and the line of research that studies these barriers to efficient agreements Transaction Cost Economics. The main assumption is that negotiations will be structured to minimise transaction costs, thus allowing parties to get as close as possible to an efficient bargaining solution. “Law and Economics” is sometime characterised as the research agenda of working out the implications of the Coase Theorem for legal theory. In this view, the appropriate role for the law (and in particular for judicial decision making) is to approximate the outcome the parties would themselves have reached in the absence of barriers to negotiation.}

Nash proposed a specific list of reasonable properties (axioms) that bargaining solutions should satisfy. Among other things, Nash argued that the solution should satisfy

- Efficiency
- Symmetry
- Invariability with respect to certain features of the environment, such as
  - the presence or absence of alternatives no party prefers
  - different but equivalent ways of writing preferences.

Nash showed that his short list of plausible restrictions leads to one specific outcome, the so-called Nash Bargaining Solution. This solution is also mathematically convenient. It can be described geometrically, and is the solution to a simple maximisation problem, which accounts in part for its popularity. Although the Nash Solution is easy to compute, it isn’t always interpreted correctly. In particular, care must be taken to distinguish between the value that players derive during bargaining from that they get if bargaining breaks down.
If the two bargainers playing divide the dollar are identical (have the same preferences, and get the same payoff in the disagreement point), the Nash Bargaining Solution divides the dollar equally.

Other authors have suggested alternative axioms and derived alternative solutions. Sometimes these axioms are justified on the basis of fairness.

3. Coalitions

Coalitions are groups of people who form an alliance to achieve a mutually beneficial goal. Non-cooperative game theory tests the stability of choices: the question is whether any individual can improve his or her lot by changing strategies. Cooperative game theory tests the stability of coalitions: the question is whether some superior coalition can form. The focus remains individual payoffs, but we now allow for collective action. If utility is transferable, all that matters is the total value of a coalition, because when this is largest, some transfers will make everyone better off.

A coalition game with transferable utility consists of
- a set of players of size $N$
- a function $\nu(S)$ that assigns a total payoff to each subset of players $S$; this is called the “worth” of $S$

Trivial games have the worth of coalitions equal to the total of what each player can achieve on their own. In these games, coalitions are not interesting. In a simple game the worth of every coalition is either zero or one. We will typically focus on the case has the worth at its largest when the coalition contains all players and the worth of any coalition independent of actions taken by non-members. The coalition of all the players is called the Grand Coalition.

Useful Definitions

- A list of the payoffs received by the players is called a profile.
- An “$S$-feasible division” of a coalition’s total payoff is a list of individual payoffs for the players that in total do not exceed $\nu(S)$.

We will restrict attentions to payoff profiles for which no player gets less than they could get acting alone and the total value of all payoffs is as high as possible, so equals the worth of the game. Such profiles are sometimes called “imputations”.

Three player divide the dollar
Consider a three player version of divide the dollar. Assume that if all three of the players can agree on a split, they get the whole dollar. If only two agree, they get $c$ cents, and alone each individual can get nothing.

Then we could let $N = \{1,2,3\}$ and define

- $v(N) = 1$
- $v(S) = c$ for $S = \{(1,2),(1,3),(2,3)\}$
- $v(S) = 0$ for $S = \{(1),(2),(3)\}$

A profile is then any set of three numbers. This is $S$-feasible so long as it restricts the payoff to single member coalitions to be zero, the payoff to the members of two player coalitions to sum to $c$ or less, and the payoff to the grand coalition to sum to 1 or less. It is an imputation so long as no player’s payoff is less than zero, and the sum of all players’ payoffs is 1.

So, for example, $(3,4,1)$ is a payoff profile where player 1 gets 3, player 2 gets 4 and player 3 gets 1; this profile is not feasible, because the payoffs sum to more than the one dollar available for any pattern of coalitions. If the coalition structure has players 1 and 2 joining together and excluding player 3, an $S$-feasible profile in three person divide the dollar for the coalition of 1 and 2 is a pair of payoffs not exceeding $c$; while for the singleton coalition of player 3 it must equal 0. This is not an imputation as long as $c < 1$, because the total payoff is less than the worth of the game.

3. The core

Solutions to cooperative games involve some definition of stability, where coalitions form to share the benefits of cooperation. The question is whether players could break the coalition structure to form a new one where they benefit. Recall that the underlying strategies are only implicit, but that it is assumed if some feasible coalition exists, the player have the means of committing to some strategy the form it if they wish to do so.

One simple definition of stability is called the core. If a profile is in the core, no coalition can be formed that makes of its members better off. The idea something akin to Nash Equilibrium: players can “defect” from a proposed coalition structure if they can form a better alternative coalition. But, to stress the point again, it is different in two crucial respects. First, the core refers to the outcome, i.e. the profile of payoffs, not the strategies. Second, it allows coalitions to defect together, taking it for granted that they have some means of enforcing any agreement that is mutually beneficial.

More formally, in a transferable utility game, the core is the set profiles $(x_i)_{i \in N}$ such that for no coalitions $S$ and $S$-feasible list of payoffs $(y_i)_{i \in S}$ for which $y_i > x_i$ for all $i \in S$. In words, for profiles in the core no coalitions can be formed that make all of its members better off.
The “alternative” profile proposed need not itself be in the core; it simply has to be feasible. If any feasible re-division can be suggested that make the members of some coalition better off, then the original suggestion is not in the core.

**Three player divide the dollar**

Consider the three-player divide the dollar game defined above.

1. Since individual players can get zero on their own, the core must include only non-negative payoffs. That is, since \( v(\{i\}) = 0 \) a core allocation must have non-negative payoffs. Otherwise, some individual player can do better on their own (in a “coalition” of 1) than in the proposed allocation. This player will refuse to participate.

2. The core also requires efficiency: \( x(N) = 1 \). Otherwise some coalition \( S \) can form to increase payoffs to this amount.

3. Finally, in the core, no coalitions of two players can form to increase their collective payoffs: i.e.

   \[ x(S) \geq c \text{ for } S = \{1,2\} \text{ or } \{1,3\} \text{ or } \{2,3\}. \]

   That is, since the combination of any two members can always get \( c \), the total payoff in the core for any of the three possible two player coalitions must be at least as large as this, or some two player coalition can form to improve member payoffs.

You should convince yourself that if \( c > 2/3 \) no division of the payoffs can satisfy all of these conditions. This is because no matter what the proposed allocation is, two players can always do better by teaming up to cut out the third. In this case, the core is said to be “empty”. In contrast, if \( c \leq 2/3 \) any allocation that offers every player at least one half of \( c \) is in the core.

Three player divide the dollar has an empty core when \( c \) is too large, but many other games do have profiles in their core. It is possible to find technical restrictions on the structure of the game that ensure that the core is not empty. Game theorists have also defined other solution concepts. These solutions are typically more restrictive in what they allow players to “propose” as alternatives. Recall that the core requires that no feasible alternative be better any subset of players. But the feasible profile might itself be outside the core, so also break down. Alternative solutions rule out some kinds of “objections” to a profile.
4. Shapley Value

The Shapley Value can be thought of as a profile of payoffs that survives certain specific objections. These objections are challenges by one player of the payoff allocated to another player. An objection is a legitimate ground for reallocation of value, and so upsets the proposed profile, unless it can be met with a specific counter-objection in response.

There are two kinds of objection, each with an associated counter-objection.

**Objection 1:** I deserve more, because if I leave, you will lose some of your payoff.

**Response 1:** That isn’t a legitimate reason for changing payoffs, because if I leave, you will lose at least as much than I lose when you leave.

**Objection 2:** I deserve more, because I can persuade others to exclude you, and I will gain.

**Response 2:** Not legitimate: I can exclude you, and I gain at least as much.

The Shapley Value has the property that every objection of these two types can be met with the associated response.

If you think about these, you will see that each statement refers to an alternative allocation, but doesn’t explicitly say what allocation. If I leave then…what? The answer is: the Shapley Value, now for a game with one fewer players (either due to departure from objection 1 or exclusion from objection 2). So these objections and counters are self-referential (or more accurately, recursive). To work out whether an objection is valid, you need to keep digging further, asking what the Shapley Value is for successively smaller games. Eventually, this process must end, as the number of players is finite.

**Computing the Shapley Value**

Rather than testing a profile to see if it survives repeated rounds of objections as above, we can compute the Shapley Value by considering the marginal contribution of players. That is, work up from the bottom adding players to get to the grand coalition.

This involves a (potentially very large, but sometimes in practice feasible) “combinatoric” exercise, since the building up over coalitions can proceed in a number of orders.

1. Compose the set of players into coalitions of every possible order. For example, when \( N = 3 \), the “grand coalition” can be formed in six ways:
   
   \( (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1) \).
Think of these as growing with the addition of players. So the third one, for example, starts with player 2, who is joined by player 1 and then finally joined by player 3. In the end, each of these contains all the players.

2. For each of these orderings, we can compute the marginal contribution of players as they join the coalition. So for example, again looking at the third in the list, (2,1,3), we can determine what is the worth of player 2 acting on her own. This “stand alone” worth is player 2’s marginal contribution in this ordering. Then determine what is the worth of the coalition (2,1). The difference between this amount and player 2’s stand alone worth is the marginal contribution of player 1 in this ordering. Finally, we can determine the value of the grand coalition, (2,1,3) that we get when player 3 joins. The difference between this and the coalition of players 2 and 1 is the marginal contribution of player 3.

Notice that by construction the sum of these marginal contributions is always the value of the grand coalition, \( v(N) \).

3. For each player find the average of the marginal contributions across each of these orderings.

The average you compute for each player is the Shapley Value. Notice, since the marginal contributions sum to the value of the grand coalition for each ordering, the average across these must also sum to the value of the grand coalition. So the Shapley Value is equal to the worth of the game.

**Three player divide the dollar**

We can once again apply this solution concept to our running example. Again assume that coalitions of two players can claim the whole dollar \( c = 1 \) as can the grand coalition with all the players, while single players on their own can claim nothing. The following table indicates the marginal contributions for each player in all six of the possible orderings, and takes the average.

<table>
<thead>
<tr>
<th>Ordering</th>
<th>Player 1 Marginal</th>
<th>Player 2 Marginal</th>
<th>Player 3 Marginal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,2,3</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1,3,2</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1,2,3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2,1,3</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2,3,1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3,1,2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3,2,1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Average</td>
<td>2/6 = 1/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
</tbody>
</table>

In each case, it is the second player joining the coalition who “contributes” the dollar to its worth. The first player alone cannot make a claim, but together with the second can do so. The third player adds nothing.
So for this game, the Shapley Value is \((1/3, 1/3, 1/3)\).

*The Shapley Value and Fairness*

The Shapley Value can be shown to follow from a simple set of appealing axioms. These can be viewed in terms of “fairness” and so the Shapley Value might be seen in some sense fair. It does seem to survive some thought experiments where other simple notions of fair allocations stumble. The divide the dollar game hints at why this is so: since each player on her own is symmetrically unable to claim the dollar, and under symmetric situations each player is equally productive in joining coalitions, the outcome splits the dollar equally.

It turns out that the Shapley Value is the only imputation that satisfies three axioms.

1. Symmetry. This is what was just mentioned: if players are interchangeable, they receive the same payoff.
2. So called “Dummy” players who do not affect the total value of any coalition get their stand-alone value. (So in the example above, if a fourth player could not join in claims on the dollar (say they were a non-resident), but had $0.50 of their own, the Shapley Value would award them $0.50, as you can easily verify.)
3. Additivity. This is less intuitive, depending on the context. It links games together by requiring that the value in the two games taken together is the same as the sum of the values of the two taken separately.

These axioms can be viewed as a list of requirements for an allocation to be fair.

*The Shapley Value and Cost Sharing*

There is nothing requiring the values we have been discussing to be positive, so we could just as well be talking about how coalitions share costs rather than benefits. The “contribution” is then the additional cost of including someone in the coalition. Shapley Value is *actually used by real people* to allocate costs across project participants. It is computed by considering the costs each pay on their own, and incurred by the coalition as they join, again for all possible orderings, and then averaging across these.

Depending on the trustworthiness of the cost estimates, a process like this could feasibly be used to divide to cost of a new sewage treatment plant across municipalities in the Capital Regional District. For example, if Sooke were most efficiently served by its own stand alone system, it would contribute nothing to the costs of any coalition beyond these stand-alone costs, and this would be its “fair share”. (Sooke would be a “dummy player.”) Esquimalt, in contrast, might have some easily available land for large scale treatment plants, and is centrally located. This might mean that adding it to a coalition of, say, Saanich and Victoria, would not increase costs by much, despite the increased usage. It then would capture some of these gains and perhaps pay less than its stand-alone cost. A question below asks you to explore how this might work in practice.
Practice Questions:

1. In the three-person divide-the-dollar game example above there were some conditions under which the core is not empty. Show that when c=0.6 the payoff profile (0.3, 0.4, 0.3) is in the core, by showing that players cannot form any coalition, including those with one member, and improve their payoffs.

2. Greater Victoria is facing the prospect of building a regional sewage system. Part of the project will involve transferring raw sewage across municipalities to treatment centres. Somehow the cost of this line will be shared, and various ways of cost sharing might be imagined. One possibility is to use the Shapley Value to cover the cost. (Building the processing plant to the appropriate specifications is another question. Here we just think about the sewer lines.)

Imagine that the system and associated connection costs are as in the following diagram:

![Diagram of sewage system]

So for example, it takes $5 million dollars to link Oak Bay line into the system by joining it to the connection with Saanich.

(a) Create a table showing the marginal contribution cost of hooking up the municipalities for every possible ordering. Assume that if, for example, Saanich is hooked up, the line passes through Victoria so no additional cost is incurred to also add Victoria to the system.

(b) Use the Shapley Value to propose a cost share for each municipality.
(c) Some people might argue that since both Oak Bay and Esquimalt can join at the cost of $5 this is a fair charge for these municipalities. Does this agree with the Shapley Value assessment? Which argument do you find more convincing?

3. A job fair that has attracted two employers and one job seeker, and is the only chance to hire a worker for the summer. The worker can only work at one job. An employed worker produces $100 worth of value per day, but a worker without a job and a firm without a worker produces nothing.

(a) Is there any profile of (daily) payoffs to the three players that is in the core? Explain briefly.
(b) How does the Shapely Value distribute the output of the worker between the three players?
(c) Do you think the Shapely Value is “fair”? Why or why not.
Chapter 6: Mixed Strategies

Until now we have restricted players to “pure strategies,” where they choose a specific action at every point in the game at which they may be required to move. When players instead “randomize” over the possible actions, players are said to be using “mixed strategies”. A player using a mixed strategy decides on the probability that each pure strategy is played, and then leaves the specific choice to be determined by chance.

We study mixed strategies for three reasons. Mixed strategies are technically useful: mixed strategy equilibria exist in some games that don’t have a pure strategy equilibrium. Nash’s proof that all (well defined) games have a “Nash” equilibrium requires that we allow for mixed strategies. Second, mixed strategies are sometime realistic. We do observe players making choices with randomization. Finally, the mathematical structure of mixed strategies has another useful and interesting interpretation in evolutionary game theory.

For many students the material on mixed strategies is the most difficult part of this course. The difficulties arise in part from the algebra, which may or may not be familiar from high school math class. We saw much of this algebra when we took that Detour through Expected Utility Theory; review those notes if you get lost in the material here. However, the larger challenge is conceptual. Even people with a solid grasp of algebra need to think carefully about mixed strategies when they first encounter them. The math is much easier if you understand what we are trying to find. To grasp what’s going on, it is very helpful to find a partner and play matching pennies a few times. As you are doing this, reflect on what you’re doing, and why. Think in terms of Nash Equilibrium. Once you get this logic straight, the arithmetic will be much clearer. Below I solve Matching Pennies with algebra, so you can map the steps into your personal experience with the game.

Key terms: pure strategy, mixed strategy, mixed strategy Nash Equilibrium

1. Mixed strategies

First, some definitions: Pure strategies require players to choose a definite action at each decision node. In mixed strategies players choose the chance that any given pure strategy will be played. Then in the play of the game, actions are chosen “randomly” according to a probability distribution. The mixed strategy is a probability distribution defined on the set of pure strategies, and this is what players choose. Viewed this way, you can see that pure strategies are a special case of mixed strategies, where all of the “weight” of the probability distribution is placed on one single pure strategy.
Example. Rather than choosing to “turn left” (a pure strategy) or “turn right” (another pure strategy), a player can instead decide to toss a coin and <<“turn left” if it comes up heads and “turn right” if it comes up tails>> (a mixed strategy). The coin serves as a randomization device, which (if it’s fair) generates the probability distribution with equal likelihood of each pure strategy being chosen. Picking the coin as the randomization device is choosing the mixed strategy with equal chances of playing the two pure strategies.

A different mixed strategy would use the toss of a dice and <<“turn left” if it came up a one or two, and “turn right” otherwise>>. This mixed strategy places 1/3 weight on the pure strategy “turn left”, and 2/3 weight on the pure strategy “turn right”.

Finally, the strategy <<“turn left” if it comes up heads and “turn left” if it comes up tails>> is the mixed strategy of turning left with probability 1. This is the same as the pure strategy “turn left”. Because we can always choose probability distributions that put all the weight on a single pure strategy, pure strategies are a special case of mixed strategies.

To summarize, mixed strategies assign probabilities to a set of pure strategies. At one extreme, all the probability is assigned to a single strategy. In this (special and trivial) case, the mixed strategy is really just a pure strategy. Typically, when we refer to mixed strategies we mean that more than one pure strategy has a positive probability of being played. It is possible to have some pure strategies played with probability zero while others are played with positive probabilities. A completely mixed strategy places some positive probability on all pure strategies available to a player. Most of the examples we will explore are one shot games with two pure strategies, and these have infinitely many (completely) mixed strategies, as the probabilities can be any real numbers that add to one.

2. Why would a player be willing to randomize?

In most of the games we’ve studied so far the best response to a pure strategy is a pure strategy. For example, for player 1 in Matching Pennies (who wins on a match) the best response to Heads is Heads and the best response to Tails is Tails. Since players will choose best responses, to use a mixed strategy it must be that the best response is an assignment of probabilities to pure strategies. Why ever would a player randomize? It turns out that by thinking about this question you can gain insight into how to compute mixed strategy equilibria. So think about it carefully. (Better still, play matching pennies while you do so, and think carefully about what you are doing, and why.)
The only time a rational player is willing to randomize over two or more strategies is when she is \textit{indifferent} between them. \textit{This is the crucial idea}. If you get this, you’ll understand the algebra to follow; if you don’t, you won’t.

So suppose this weren’t true. Suppose you told your friend “I’m going choose my route with a mixed strategy, based on the toss of a coin: if the coin comes up head, I will turn left at the next intersection; if it comes up a tail, I will instead turn right.” Then imagine that the coin comes up head, and you think “To heck with turning left, I live to the right!” Then, despite what you told your friend, you are not willing to play the mixed strategy. Why? \textit{Because you strictly prefer one of your pure strategies}.

Using a mixed strategy requires a player to delegate the choice of their pure strategy to some randomization device, like a coin toss. A rational player will be willing to delegate the choice of a pure strategy only when she has \textit{no strict preference} between the strategies that might be chosen by the device. Otherwise, she will prefer to make the choice herself.

Once you understand this idea, how we work out mixed strategy best responses will make sense. Think about how you play matching pennies. Do you randomize? Why? Are you indifferent to heads or tails? Why? Is it because of something you do, or something the other player is doing? When wouldn’t you be indifferent?

3. Calculating mixed strategy best responses

I just argued that, given the anticipated strategies chosen by other players, a player must be indifferent between all of the strategies she plays with positive probabilities in a mixed strategy. This leads to the startling result that, \textit{from the perspective of the player choosing a mixed strategy}, the exact probabilities chosen do not matter. But the probability distribution is the mixed strategy. This is the counter-intuitive part of this whole exercise: equilibrium will require players to use a particular mixed strategy, but if you are willing to play one mixed strategy, you are willing to play many different mixed strategies.

For example, assume a player is indifferent between going left and right at the next intersection. Then she is indifferent between going left with 1/3 probability and right with 2/3 probability, and also between going left with ½ probability and right with ½ probability, and, in fact, simply going left for sure. All these are the same to her, since she does not care whether she goes left or right. This is the very essence of indifference.
Willingness to randomize requires indifference, and the indifference to outcomes extends to indifference over the exact probability distribution chosen: if one mixed strategy is a best response, so are others.

We don’t seem to be getting very close to understanding how players choose their mixed strategies. They all seem equally appealing. On the other hand, we might expect that rational players do what they do for a reason. What is the motive for randomizing, and in particular choosing any one particular probability distribution? In fact, there may or may not be a motive. Sometimes there is, and these examples are more intuitive. But they isn’t always a motive, and even in games where players benefit from being predictable, completely mixed strategy equilibria may still exist.²²

So it’s a matter of definition whether or not a completely mixed strategy equilibrium exists, and ascribing motives (beyond the desire to get the best payoff) is not necessary for the concept to make sense, at least formally. Nevertheless, thinking about motives can help get the idea. So let’s start by assuming that players have a simple motive: they choose probabilities “to keep the other player guessing”. This motive makes sense in many cases, for example, Matching Pennies and many other zero-sum games, such as professional sports.²³ If this goal is achieved, then the probability distribution chosen by a player will make the player’s opponent indifferent between at least some of her pure strategies. If instead she had a unique best response to the player’s strategy, it is as if she’s got the player “figured out,” and knows what to do. In zero sum games, such as Matching Pennies, she could then use this knowledge to the player’s disadvantage.

4. Mixed strategy Nash equilibrium

If players are successful in keeping other players guessing, then they will choose the probabilities in their mixed strategies to make other players indifferent between some (or all) of their strategies.²⁴ If these other players are indifferent, then they too are willing to play mixed strategies. For this to be an equilibrium, it must be that every player chooses probabilities in such a way that the others are willing to “mix”, i.e. randomize their own choice. When this is true, the resulting set of probability distributions over pure strategies is a mixed strategy Nash Equilibrium.

---

²² Since pure strategies are a special case of mixed strategies, all the games we consider with pure strategy equilibria have mixed strategy equilibria, even ones with dominant strategies.
²³ Typical examples include running or passing in football, returning forehand or backhand in tennis, and bluffing in poker. Even if everyone knows your hockey team has the league’s best centre, a unbreakable rule to passing to that player on every three on two is unwise, since the defense can be prepared. A question at the end of the lecture asks you to solve for some “undesirable” mixed strategy equilibria.
²⁴ With two strategies as in most of the examples we look at, “some” is all.
Note the direction of causality here: player 1’s choice of probabilities makes player 2 indifferent between some (or all) of her pure strategies, hence makes her willing to mix. Then, as we’ve seen, she will be indifferent to exactly which probability distribution she uses. But for it to be an equilibrium, it does matter what player 2 chooses: player 2 must choose her mix strategies in such a way to makes player 1 willing to choose a mixed strategy.

Consider the Matching Pennies. Most people playing this quickly learn to play the mixed strategy equilibrium, which I will construct formally in the next section of these notes. But first, think about the logic. When you played, after a couple of rounds, you probably just tossed the coin on the table. Many people play thinking that it doesn’t matter what they do (and maybe even that they aren’t “choosing a strategy” at all, but that’s before they knew about mixed strategies!)

One reason you might think it doesn’t matter is that you see your opponent choosing to randomize by tossing the coin on the table. Given that you have no reason to care. (Alternatively you might have no good guess as to what they are going to do; experience might improve your prediction about their choice, unless, of course, they are just randomizing.) Conditional on your opponent randomizing, you don’t care what you do. If you were confident they would continue to toss the coin, you could play heads every time and do just as well (say if you were playing against a computer programmed to play heads 50% of the time). But when playing against a rational player, doing something other than simply tossing the coin wouldn’t be part of an equilibrium.

In fact, suppose you are player 2 and your coin was not fair, but instead was weighted in a way to make heads come up ¾ of the time. If both you and your opponent knew this, and you simply tossed the coin, your opponent would be better off choosing heads every time. Why? By always playing a head when you toss your biased coin, they win ¾ of the time. (If they had a fair coin and tossed it, they win only ½ the time.) Of course, if player 1 decided to do that, to play heads for sure, player 2’s best response would not be to toss the coin, but to play tails! Player 1’s choice is then not a best response, and so on.... In matching pennies the unique Nash Equilibrium has each player choose heads with probability ½. If you have a biased coin, you will need to find a fair one to help you choose your strategy, because tossing it won’t be ensure you win your one half of the trials. One reason Matching Pennies players equilibrium mixed strategies is that the penny itself is designed perfectly to generate the equilibrium probabilities.

5. Matching Pennies solved formally

Now we will bring in the heavy machinery to formally solve Matching Pennies. This involves algebra, but the logic is exactly as we’ve just explored verbally. This method can be used solve all 2X2 games for mixed strategies. The method involves several steps. First, we assume that player 2 is choosing according to some randomization. Don’t ask why; we’ll get to that in a moment. Instead, ask what must those probabilities be to ensure that player 1 is willing to randomize over the choice of heads or tails. We work
this out by calculating the expected utility that player 1 gets from each pure strategy (H and T), then finding the specific probabilities (used by player 2) that make these expected utilities for player 1 equal. Why equal: because that is what indifference means to players who obey the theory of Expected Utility. With the probabilities for player 2 derived from this step, we know player 1 is indifferent between each pure strategy. This means that player one is herself willing randomize. Now we have a why, at least for player 1, and it’s based on player two doing the right thing.

Then we reverse the process: we assume that 1 is using some mixed strategy probabilities and compute what these probabilities must be to make player 2 indifferent. Here, however, the assumption that player 1 randomizes is justified by the assumption that player 2 is playing the mixed strategy we just calculated. With these probabilities for player 1’s strategy, we can justify player 2’s original mixed strategy used to make player 1 indifferent. (If this all seems a bit circular, that’s because it is! But so are all Nash Equilibria, in that one player’s choice justifies the choice of the other, which in turn justifies the choice of the first. This is the “mutual” in mutual best responses.)

In full detail, here’s what we do:

**Step 1:** Assign some “unknown” probabilities to each of player 2’s pure strategies. I will add labels to the normal form to keep track of this assignment.

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Player 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>1-p</td>
</tr>
<tr>
<td>Head</td>
<td>1-p</td>
</tr>
<tr>
<td>Tail</td>
<td>1-p</td>
</tr>
</tbody>
</table>

I’ve added them for both players. The assignment is interpreted as player 2 chooses head with probability $p$ and tail with probability $1-p$. (Recall these must add to 1 so we only need one letter to represent all possible ways that 2 could ever choose to make this choice.) There is nothing special about the letters $p$ and $q$: they are just place holders for the probability values we are trying to calculate.

**Step 2:** Given this assumed strategy for player 2, we now calculate the expected utility for each of player 1’s pure strategies. The expected utility for a pure strategy is found by multiplying the utility of each outcome (i.e. the payoff) by the probability of the outcome, and then summing over all possible outcomes. When player 1 chooses Heads there are two possible outcomes: with probability $p$ player 2 also plays Head and player one wins (utility = 1) and with probability $(1-p)$ player 2 plays Tail and player 1 loses (utility = -1). This means the Expected Utility for Player 1 of playing Heads is
This makes sense: if player 2 plays Head for sure \((p = 1)\), Player 1’s payoff from Heads is

\[
EU_1(H) = p \times 1 + (1 - p) \times (-1) = 2p - 1
\]

If player 2 plays tails for sure, player 1 gets a payoff from Head of -1. As \(p\) varies between zero and one, the expected utility for player 1 choosing the pure strategy Head also varies, between -1 and 1.

We need to do the same thing for Player 1’s choice of Tail. In this case, Player 1 loses when 2 plays Head, which occurs probability \(p\), and Player 1 wins when 2 plays Tail, which occurs with probability \(1 - p\). The Expected Utility is

\[
EU_1(T) = p \times (-1) + (1 - p) \times 1 = 1 - 2p
\]

**Step 3:** In general, for any given value of \(p\), player 1 will prefer to choose either Head or Tail. For Player 1 to be willing to play a mixed strategy, it must be that Head and Tails are equally good pure strategy choices; this requires that they have the same expected utility. That is, it must be that

\[
EU_1(H) = EU_1(T)
\]

This implies that

\[
2p - 1 = 1 - 2p
\]

or

\[
p = 1/2
\]

We have determined that if and only if Player 2 is equally likely to choose Head as to choose Tail, then Player 1 is willing to randomize; when Player 2 is doing this, Player 1 is indifferent between her own choices. Of course, so far we’ve done nothing to ensure Player 2 will play this mixed strategy: 2’s preferences have not played any role in working out the probability that he plays Head or Tail. The requirement for \(p = 1/2\) is entirely to lay the ground for Player 1 to choose a mixed strategy.

**Step 4:** Since Player 1 is indifferent between Head and Tail (conditional on \(p = ½\)), Player 1 is willing to choose any mix of Head and Tail: i.e., she is willing to choose Head with probability \(q\) for any value of \(q\). Next we need to figure out what \(q\) must be to validate the assumption in step 1 that Player 2 was willing to randomize. You can see that this problem is so symmetric that it isn’t surprising that the \(q\) we are looking for is also \(q = ½\). (This is not generally true.)

Repeating steps 2 and 3 without the commentary, we see that
Step 5: Now we are done. We have constructed a mixed strategy Nash Equilibrium: the (mixed) strategy <<Play Head with probability ½ and Tail with probability ½>> for player 1 is a best response to the strategy <<Play Head with probability ½ and Tail with probability ½>> for player 2, and vice versa. These are mutual best responses.

In any one play, anything could happen. If play is repeated for a long time, each player will win one half of the games. On average, their gains equal their losses, and so the average payoff is zero. (Not much fun. I expect you discovered this yourself when you played it, which is why I keep insisting you should. I admit it, the game is boring.) We can calculate this directly by figuring out the Expected Utilities from the formulas above. For example, for player 1, by substitution in $p = ½$

$$EU_1(H) = 2 \times \frac{1}{2} - 1 = 1 - 1 \times \frac{1}{2} = 0$$

6. Graphing Best Response Functions

To visualize the mixed strategy equilibrium, and see exactly how these probabilities are “mutual best responses” we can graph the players’ best responses as “functions” of the probabilities in each other’s mixed strategies. The “function” we are looking for is a relationship between $p$ and $q$. For Player 1, a best response function takes as its input the probability that Player 2 plays Head, and gives as output the probability of also choosing Head that leads to the highest (expected) payoff for 1. Since we have (arbitrarily) assigned the letters $p$ and $q$ to these probabilities, Player 1’s best response will take a value of $p$ and produce a value (or range of values) of $q$. Player 2’s best response does the reverse, taking a value of $q$ and returning a value (or range of values) of $p$.

In Matching Pennies, there are three possibilities for the best response to any probability chosen by the opposing player: Head is a best response, Tail is a best response, or both Head and Tail are best responses. We just calculated when the third would be true, and found it is true only if $p = ½$. Otherwise, one and only one of Head or Tail will be a best response. (Again, think of your own play of this game.)

---

25 In case you are finicky about such matters, I will point out that these are actually correspondences.
Player 1

Player 1’s best response can be shown on a graph as follows. The horizontal axis is the probability that Player 2 plays Head (i.e. \( p \)); the vertical axis is the “best response” by Player 1, which is the probability of playing Head (i.e. \( q \)). That is, when Head is a best response, Player 1 should choose \( q = 1 \) as her mixed strategy (which is the pure strategy Head). When instead Tail is her best response, she chooses the mixed strategy \( q = 0 \). When either Head or Tail is a best response (i.e. when \( p = 1/2 \)), then any \( q \) between 0 and 1 is equally good, and so all of these values are best responses.

The picture:

The bottom horizontal axis indicates the probability with which player 2 chooses Head. The left vertical axis indicates the probability that player 1 chooses Head. The blue line is player 1’s best response.

- When \( p \) is less than \( \frac{1}{2} \), player 2 is more likely to play Tail than Head, and player 1 should always play Tail. Therefore 1’s best response sets the probability of Head to be zero. In this range \( EU_1(T) > EU_1(H) \).
- At \( p = \frac{1}{2} \) any choice of probabilities for player 1 is as good as any other because \( EU_1(H) = EU_1(T) \). The best response line is vertical: any point on the left axis is a best response to \( p = \frac{1}{2} \).
- For \( p \) greater than \( \frac{1}{2} \), where player 2 is more likely to play a Head, player 1’s best response is to play a Head for sure: \( EU_1(H) > EU_1(T) \). This requires \( q = 1 \).

Make sure you understand this logic, and how these statements are consistent with the equations above that define Player 1’s Expected Utility as a function of Player 2’s choice of probability \( p \).
Player 2

We can do the same thing for Player 2’s choice of a best response to any mixed strategy $q$ of Player 1. In fact, we could draw an identical picture, just switching the $q$ and $p$ (and the label on the best response). It is more convenient to do this in a way that allows the pictures to be combined. This is done by graphing Player 2’s best response from the vertical to the horizontal axis. The combined picture is:

![Best Response Diagram]

Notice that these two best response lines intersect when $p=1/2$ and $q=1/2$: this makes sense, since this is when (and only when) the players are playing mutual best responses. This is the unique Nash Equilibrium for the game. You can also see that if one player fails to play the specific probability called for by the equilibrium, the opponent suddenly is no longer indifferent, and will instead prefer one of the pure strategies. Finally you can see that there is no pure strategy equilibrium, because the best responses do not intersect at any of the corners (which are the points representing both players choosing pure strategies).

7. Mixed and Pure Strategy Equilibria coexist

As a general rule, if there are multiple pure strategy Nash equilibria, there will be mixed strategy equilibria that are randomizations over the pure strategies involved in the pure strategy equilibria. For example, consider the Driving Game with the normal form

<table>
<thead>
<tr>
<th></th>
<th>Driver 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>2,2</td>
</tr>
<tr>
<td>Right</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Earlier we saw that this game has two pure strategy Nash equilibria (Left, Left) and (Right, Right). You should repeat the steps in section 5 above to convince yourself that this game also has a mixed strategy Nash Equilibrium where each driver chooses sides of the road randomly. (This is one of the exercises at the end of the lecture.) If you
construct the best response functions as above, you will find that, unlike those for Matching Pennies, they intersect three times. The two in the corner represent the pure strategy equilibria and the one in the middle represents the mixed strategy equilibrium.

The Driving Game is unlike Matching Pennies in another way: the players have no desire to deceive each other. Notice that the Expected Utility in the mixed strategy equilibrium is lower than that in either of the pure strategy equilibrium (because sometimes the drivers crash!). Our suggested motivation that players are “trying to keep each other guessing” is missing. Here the equilibrium must be sustained by some other source of beliefs, and drivers would work hard to find ways to clear up the confusion. What that source of beliefs would be is hard to imagine in many cases, and it seems unlikely that we would observe the knife-edge outcome of a mixed strategy equilibrium occurring in any real world game of pure coordination. But the logical possibility is there, and the fact that players wish to coordinate on one of the two pure strategy equilibria does not ensure that they are able to do so.

8. Nature as a player

Players sometimes face uncertainty that they do create by the use of mixed strategies. The simplest way that randomness can enter a game is exogenously: that is in a way unaffected by the choices of any player. (The “random” location of the prize in the Monte Hall game is an example of this.) This kind of randomness is typically modeled by including “nature” as a special player in the game. Nature gets a move, but is indifferent between all the outcomes of the game, so nature has no preferences.

Practice Questions:

1. The following three normal form games can be solved by the method used above to find the mixed strategy equilibrium in Matching Pennies.

   (i)  
<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>2,1</td>
</tr>
<tr>
<td>Down</td>
<td>-1,1</td>
</tr>
</tbody>
</table>

   (ii)  
<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>2,1</td>
</tr>
<tr>
<td>Down</td>
<td>-2,1</td>
</tr>
</tbody>
</table>

   (iii)  
<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>3,1</td>
</tr>
<tr>
<td>Down</td>
<td>-3,1</td>
</tr>
</tbody>
</table>

   For each game,
   (a) Confirm that it does not have a pure strategy Nash equilibrium.
   (b) Find the Mixed Strategy Nash Equilibrium.
   (c) Graph the best response functions for both players. Circle the Nash Equilibrium.
(d) Do the players in games (ii) and (iii) have the same ordinal preferences? Do the games have the same equilibrium? Explain why or why not.

2. Find the mixed strategy equilibrium for the games of the driving game, chicken, battle of the sexes and stag hunt described in Lecture 2. Are players better off, worse off, or equally well off in the mixed strategy equilibrium as they are in the pure strategy equilibria?

3. Does the prisoner’s dilemma game have a mixed strategy equilibrium? Why or why not? Carefully examine the algebra.

4. Rock, Paper, Scissors is a 2X3 variant of matching pennies I assume you are familiar with. Write a strategic form for this game and show that it has no pure strategy equilibrium. Calculate its mixed strategy equilibrium. Is it possible to “win” this game? (Search Google to discover that a robot created at the University of Tokyo always wins at Rock, Paper, Scissors. Explain why a pedantic game theorist like your instructor would dispute this claim.)
Chapter 7 Multiplayer Games

To this point in the course, most of our examples have been two player games. Strategic situations frequently involve more players, sometimes many more. In general we speak of multiplayer or "N player" games, where N stands for any integer, 2, 3, 4,...to some potentially very large (but finite) number. Increasing the number of players adds realism but complicates the models. Nash equilibrium requires every player to choose a best response to the choices of all the other players. As the number of players gets large it rapidly becomes impossible to consider each possible combination of strategies and test whether any player can benefit from switching strategies. Other means of solving the games must be found.

In this lecture we consider methods of solving multiplayer games. After introducing the key techniques several examples are presented.

Key terms: N-person game; state variable; representative agent; proportional game

1. Simplifying assumptions revisited

All models require simplification: that is the value of building models. Well designed models simplify away the inessential to present with clarity the essential details of the situation being represented. Restricting attention to two players is one such simplification; clearly most social situations have more than two persons involved. By reducing the dimension of the problem, however, we can easily examine the strategic interactions, and find solutions. But this simplification isn’t always appropriate, and precludes some interesting situations. Happily, in many of these cases other simplifications will also permit straightforward analysis. That is, we aren’t getting away from the need to simplify, but instead substitute one simplification for another. (Alternatively, we could find more powerful solution methods to deal head on with the added complexity; an example of this for multiplayer games would be the use of numerical simulations methods.)

Below we study multiplayer games under two (complementary) simplifying restrictions. We will assume that

1. The players are symmetric: each has the same choice of strategies, and symmetric preferences over outcomes;
2. The only thing that determines which choice is better is the number of players who choose each strategy.

These assumptions are so frequently useful that we will identify them with special labels. Assumption 1 is the **representative agent** assumption: one only need consider a “typical player” to learn everything there is to know about the game. Since everyone is symmetric, facing the same choices and viewing the world the same way, once we’ve found the best responses of one player we’ve found them for all players. Note that by symmetric preferences over outcomes, we are *not* assuming that players agree, but rather that players only differ by an arbitrary index. Many of the examples in previous lectures satisfy this restriction. For example, in the game of Chicken, each player prefers to not swerve when the other swerves, to both swerving, to swerving when the other does not, to neither swerving. The best responses for player one are exactly those for player 2.

The second assumption requires best responses to depend on a single **state variable**: a summarizing statistic that tells players everything they need to know about the game. The "state" or sometime "state of the world" refers to a list of information that summarizes everything that a player needs to know about the world to choose a best response. In general these can be very complicated, but sometimes a small number of pieces of information summarize everything that you need to know. Assumption 2 says that this state variable is the number of players choosing each strategy. For two-player Chicken, the state variable is the strategy chosen by the other player.

### 2. The traffic game

Twenty commuters must choose between taking a train and driving their cars. All they care about is the length of time it takes to get to work. The train takes 40 minutes no matter how many passengers are riding, but the time it takes to drive depends on how many commuters choose to drive. Assume that payoffs are equal to the negative of the commuting time. The payoff for the train is -40. The payoff for driving is given in the following table:

<table>
<thead>
<tr>
<th>Cars</th>
<th>Payoff</th>
<th>Cars</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-22.5</td>
<td>11</td>
<td>-47.5</td>
</tr>
<tr>
<td>2</td>
<td>-25</td>
<td>12</td>
<td>-50</td>
</tr>
<tr>
<td>3</td>
<td>-27.5</td>
<td>13</td>
<td>-52.5</td>
</tr>
</tbody>
</table>
A simple way to find an equilibrium is to test some allocations between driving and riding the train to see if any player would wish to switch behavior. Since there are a lot of potential allocations -- that is, ways that the commuters can be divided between drivers and bus riders -- there are a lot of possibilities to consider. But this example satisfies the simplifying assumptions above. First, all players view the game symmetrically. Second, payoffs depend only on the number of drivers, so we can use this as a state variable. (You should be able to see that we could instead use the number of train riders).

Let’s test the situation where two players choose to drive and the rest to ride the train. Is this a Nash Equilibrium? Consider first a representative driver: this player gets a payoff of -25 (i.e. it takes 25 minutes to get to work for the driver) versus -40 if she switches to riding the train. So driving is a best response. (This is clearly true for both of the drivers.) Now consider a representative train rider. This player would prefer to change strategies: becoming the third driver gives a payoff of -27.5 which is better than -40. As the representative train rider is in no way special, we see that all the train riders would prefer to drive. Remember that Nash equilibrium takes other players’ choices as fixed, so the representative doesn’t worry that if he wants to switch so will everyone else. The conclusion we draw is simply that the state where two players drive is not a Nash Equilibrium. Some of the players (here all of the train riders) would prefer to deviate and choose drive instead.

We could keep testing allocations of the players between driving and riding the train. But the logic of the case just considered, makes it easy to see what is needed, and that it is a Nash Equilibrium for eight players to drive. Neither drivers nor train riders can unilaterally improve their payoff by switching to the other mode of travel. A representative player is indifferent between the two strategies, each of which takes the same commuting time.

Notice that this Nash Equilibrium is not unique, in that it doesn’t say which players drive and which ride the train. The Nash Equilibrium “characterizes” the situation, showing
the share of drivers and riders, so tells us something of interest, but doesn’t say exactly “what happens”.

**Social Efficiency**

If all you cared about were the preferences of the players, you would hope that the total commuting time was as low as feasible. Notice that this is not true in the Nash Equilibrium. Since commuters do not account for the cost they impose on other drivers when they take the car, their privately optimal decisions waste time overall. If instead we were able to persuade four more commuters to ride the train, the total time commuting would fall. This would be a Pareto improvement over the Nash equilibrium: the four “extra” train riders would still take 40 minutes to get to work as they do when they drive in the Nash Equilibrium, and the remaining drivers would all get to work 10 minutes earlier. Getting a fifth passenger off the train would not reduce total commute time, as this would only save the three remaining drivers a total of 7.5 minutes, and cost the 17th train rider 10 minute, for an increase in total time of 2.5 minutes.

The implication of this inefficiency is startling. A cleverly designed tax (or road toll) could in principle raise money without reducing welfare at the Nash equilibrium. By charging a toll equivalent to ten minutes commuting time payoffs for driving would now be (- time – toll= -time-10):

<table>
<thead>
<tr>
<th>Cars</th>
<th>Payoff</th>
<th>Cars</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-32.5</td>
<td>11</td>
<td>-57.5</td>
</tr>
<tr>
<td>2</td>
<td>-35</td>
<td>12</td>
<td>-60</td>
</tr>
<tr>
<td>3</td>
<td>-37.5</td>
<td>13</td>
<td>-62.5</td>
</tr>
<tr>
<td>4</td>
<td>-40</td>
<td>14</td>
<td>-65</td>
</tr>
<tr>
<td>5</td>
<td>-42.5</td>
<td>15</td>
<td>-67.5</td>
</tr>
<tr>
<td>6</td>
<td>-45</td>
<td>16</td>
<td>-70</td>
</tr>
<tr>
<td>7</td>
<td>-47.5</td>
<td>17</td>
<td>-72.5</td>
</tr>
<tr>
<td>8</td>
<td>-50</td>
<td>18</td>
<td>-75</td>
</tr>
<tr>
<td>9</td>
<td>-52.5</td>
<td>19</td>
<td>-77.5</td>
</tr>
<tr>
<td>10</td>
<td>-55</td>
<td>20</td>
<td>-80</td>
</tr>
</tbody>
</table>

Now four drivers and 16 train riders is a Nash equilibrium. The total payoff to all commuters inclusive of the toll is still -800, as it was without the toll. However, forty “minutes” worth of toll revenue is available to the government. So long as it doesn’t affect the relative payoff to driving versus riding the train, this revenue can be distributed to players in a way that makes everyone better off (for example, each player could be given as a lump sum 2 minutes worth of the toll).
3. Auctions

The word auction comes from the Latin for increase. Auctions refer to mechanisms for buying and selling that allow “bidders” to influence the price of the object or service for sale and the chance that they “win” and get anything at all. Auctions are a very old method of selling that continues to be used very widely today. On-line auctions such as those operated by eBay and Amazon appear to be very modern, but are the direct descendents of the most ancient institutions, and can be analyzed in the same manner.

The many forms of auctions can be classed under a few headings. Is the auction operated by buyers (like a city government asking for bids to repair a street) or sellers (like the person selling her grandmother’s art collection)? The answer to this question isn’t very important: for most purposes we can treat buying and selling symmetrically. We will always talk about auctions where the seller is running the show and buyers bid. Similarly, our models will not differentiate over whether it is an object or a service that is for sale. This also can matter (since reselling objects is generally easier than reselling services) but we will always talk about selling an object.

A more important question regards whether only one or many objects are for sale, and whether they are auctioned off at once or in sequence. We only consider auctions where a single unit of the good up for sale. We ignore resale or partitioning the object between bidders and assume that one and only one bidder can win the object. We also assume that the winner is the bidder who bids the highest (which isn’t always true).

It also matters what the buyers and sellers know about the value of the object to themselves and the other players. If everyone knows everything, the seller could just negotiate with the one buyer who values the good most highly. If no one knows anything, it will be hard for them to decide what to do (should the seller even sell if she doesn’t know what the object is worth to her). For the most part we will assume that the seller and all bidders (i.e. potential buyers) know their own personal value for the object, and that everyone knows the distribution of other players’ valuations, but not the exact values others place on the object. Generally, we will assume that these values are independent and private in the sense that learning what someone else values the object at doesn’t tell you anything about your own value (or vice versa). This assumption is the simplest to work with, but not always realistic. For example, if the painting for sale could be re-sold, it might be that every bidder values it primarily for that possibility. Then it would “really” be worth the same amount to everyone — a so called common value auction.
Finally, auctions are markets that have been designed explicitly. The person organizing an auction is setting the rules of the game, and can influence all the key elements: the set of players, the strategies available, and the outcomes from any particular combination of strategies. This game is a “mechanism” for achieving some end, and the usual assumption is that a seller will choose an auction that makes revenue as high as possible. Efficiency — ensuring that the object goes to the person who values it most — is of course also of interest, but sellers will typically only care about this if it coincides with increasing revenue. There is good reason to suspect it will, as the person who values an object most highly is also likely to be the one with the greatest willingness to pay\(^\text{26}\). But this isn’t always so, and other factors sometimes supervene. For example, a “reservation price” can ensure that no sale occurs below a stated cut-off. Sometimes it is in the interest of sellers to set a reservation above their own valuation. This may mean that an object fails to sell at all, when some or even all potential buyers value it more than the seller. This is inefficient, but can lead to higher expected revenue. Here I will set aside these slightly more complex issues. Last, notice that some sellers may not be so much concerned with revenue as with efficiency or some other goal. Governments auctioning “spectrum”, frequency for cell phone signals, for example, might care about revenue, efficiency, and the implied market power in the future telecommunication market. Economists generally assume that an “optimal” auction is one that best satisfies the interest of the seller, who designs the game, and take this to be one that provides the highest revenue.

**A simple multiplayer auction game**

At first glance, writing down and solving a model of the familiar so-called “English open outcry auction” is a daunting task. After all, there are many bidders, and it isn’t clear that any particular bidder knows how many others are bidding, let alone how much these others value the object for sale (i.e. the other players’ preferences). Since bidding occurs through time, an English auction is really a dynamic game. So there is much scope for contingent strategies (for example, strategies like: if, when the price reaches $34, bidder 17 bids $35, I will bid $36.50). Working out a subgame perfect Nash Equilibrium with large numbers of bidders seems impossible.

\(^{26}\) For example we will assume that willingness to pay and ability to pay are the same: bidders are not facing a binding budget constraint.
This is made simpler when we assume that values are private and independent, and rely on the equivalence between the English and second price auctions (see the textbox below). We will study the second price auction where bidders submit written bids, the auctioneer opens them, and then awards the object to the highest bidder at a price equal to the second highest bid (the second price). The sealed bid auction does not require contingent strategies at all, as it is a static one-shot game. (Thus we know that under the assumptions here, bidders in English auctions don’t need contingent strategies either.)

### Four Basic Types of Auction

Game theorists have devised a simple taxonomy for auctioning a single unit of a good:

- **English Auctions:** “open outcry” with ascending bids. Bids start low and are raised until only one bidder remains. He wins and pays the final bid, which is only a tiny bit higher than the final bid from all other bidders.
- **Dutch Auctions:** “open outcry” with descending bids. A bid “clock” starts with a very high bid, and then falls toward zero. The first bidder who cries out and stops the clock wins and pays the price registering at that time.
- **First-price sealed bid:** bidders submit bids before a pre-specified time. All bids are then opened. The bidder who bid highest wins and pays her own bid (the first, i.e. highest, price).
- **Second-price sealed bid:** bidders submit bids before a pre-specified time. All bids are then opened. The bidder who bid highest wins and pays the second highest bid (the second highest price). These are sometimes called Vickery, after the economist who “invented” them.

From a strategic perspective (for single unit, independent private value auctions) the Dutch and first price auctions are equivalent, and the English and second price auctions are equivalent. To see this, think of the strategy as an instruction to an agent who will play the game for you. Imagine a room full of agents with these instructions at a Dutch descending bid auction. The clock will be stopped by the agent holding the highest bid (and this will be the price). The exact same outcome would have occurred if the auctioneer had gathered up the pieces of paper with the bids written on them, and then given the object to the person who submitted the highest bid (which is also the price in the first price sealed bid auction).

The English / Second price equivalency is only a little harder to see. In the open outcry English auction, all the bidders’ agents will stay “in” until the number they have been given by the bidder is reached. One by one they will drop out. Eventually, only the agents from the two bidders who gave the highest numbers will remain. Once the lower of these two is reached, bidding will stop, and he object will go to the bidder who gave her agent the highest number. The price will be equal to the second highest number (or perhaps one “bid increment” above this). The same outcome would follow if the auctioneer had simply collected the papers with the agents’ instructions, found the two highest, and awarded to object to the highest bidder at a price equal to the second highest bid.
The second price sealed bid auction game is then a one shot game. All the players have preferences that determine their value for the good. This translates into a willingness to pay for the good. The seller is assumed to value the good at zero. The normal form is

**Players:** N bidders (potential buyers) and 1 seller  
**Strategies:** Bidders: Submit a written bid (a number); Seller: agree to sell/refuse to sell  
**Payoffs:** Bidders:  
- if bid is highest of all submitted bid, payoff is the difference between the bidders valuation and the second highest bid.  
- if bid is not highest, payoff is zero.  
Seller: payoff is price: the second highest bid.

It’s easy to see that the seller has a dominant strategy to sell for any positive price, since she values the good at zero. What is less obvious is that the bidders also have a dominant strategy. They can never benefit from submitting a bid different from their valuation for the object. The argument can be depicted on a diagram.

The picture depicts three possible strategies for bidder $i$. These are labeled $b_i$. The players actual value is labeled $v_i$. For any bid it is possible that the player’s bid is not highest, in which case the bid does not affect her payoffs. The cases that matter are those in which the player wins. By bidding above her true value, the only outcome that could change is the one where the second highest bid is above her value. (If the second highest was below her value, she would still win, but she would pay the same thing she would have won had she bid $v_i$.) But in this case she pays too much, and is worse off than has she not won. Similarly, should she bid below her value, then she can only lose. The only time the low bid leads to a different outcome than bidding $v_i$ is when someone else bids between her bid and her value. In this case she loses, when she could have won by bidding her value. Together, this reasoning implies that a player can never do better than by bidding $b_i = v_i$.

Notice that players should bid their own value no matter what other players are doing. The Nash equilibrium is therefore all players bidding their values, and the seller selling the good.
The game is simple to solve because we can view it from the perspective of a representative agent (bidder $i$) and all that player needs to know to bid optimally is her own valuation (the state variable).

**Vickery-Clarke-Groves mechanism**

This second price auction is an example of a “Vickery Clarke Groves mechanism”. In particular, this auction is a game (or mechanism, as these designed games are called) that allocates the object to the highest value player by getting bidders to truthfully reveal their value. Before bidding this was private information. In the game, seller “pays” for the information about values, in the sense that the bidders expected to gain, and the winner did gain. The seller uses this information to allocate the object to the highest value player which makes surplus as large as possible, and allows the seller to get the highest possible price.

Recall that if the values had been public knowledge, the seller could have simply gone to the highest value bidder and stated a take it or leave it offer to sell for that player’s value. The player wouldn’t lose by saying yes. In the auction, this player expects to gain by participating, not just break even. The expected gains to bidders that flow from their having private information about their values is called “information rent”. How much rent they get is related to how much uncertainty there is in the player’s valuations, which is in turn related to how important it is for the object to go to the highest value player.

VCG mechanisms like the second price auction work by setting the rules up so player who don’t tell the truth about their value have their payoff fall relative to what it would be if they tell the truth. But more than that, these mechanisms are tuned so that the amount of that loss is exactly equal to the value of surplus lost due to the inefficient allocation that results from the false report. If the “lie” is harmless, the cost to the liar is also zero; if the lie results in the object being misallocated, the cost to the liar is equal to the loss from that misallocation. This means VCG mechanisms align the players’ incentives with the social goal of creating the largest possible surplus.

**Common Value Auctions and the Winner’s Curse**

Game theorists have also studied games where the players’ values are not independent and private, but are related. The simplest case is where the object is actually worth the same amount to all players (say a contract for supplying a service) but different players estimate that value differently. In common value auctions, bidders’ strategies are not so simple: in fact, they would all like to know the estimates of the other players. Interestingly, winning is bad news in a common value auction, because it means that your estimate is higher than everyone else’s. If their estimates are at all informative about the true value of the object, the fact that you guessed highest means your estimate is likely to be too high. If you don’t take this into account, you will bid too high, and suffer from the **winner’s curse**. Of course, rational bidders do take this into account, and never bid as high as they think the object is worth. By shading their bids to account for the winner’s curse, it can be avoided. In real life, inexperienced bidders sometimes

©David Scoones 2007-17
do not do this, and find themselves unhappy once they own the object and learn its true value.

4. The queuing game (McCain)

Each player chooses between lining up and sitting waiting. The payoff from standing in line depends on a player’s position in line, and is reduced by the effort required to stand rather than sit. Once the line is served, people sitting are called randomly. The Nash equilibrium has $j$ people standing in line and $N-j$ people sitting to wait. The equilibrium number $j$ in the queue is determined by the conditions:

a) that each of the $N-j$ sitters prefers to sit than become the $j+1$st in line, and

b) the $j$th person in line prefers to stay in line than to sit down and be randomly served with the $(N-j-1)$ sitters.

That is, each player chooses between two strategies, queue or sit, and in equilibrium their choice is a best response to the choices of all the other players.

Like the driving game, this is a multiplayer variant on a social dilemma: if people could coordinate, they’d be better on average if they all sat. But when everyone sits, the first person to queue benefits, so everyone sitting (the cooperative solution) is not a Nash Equilibrium.

Also notice that the equilibrium is not unique: any of the six customers can fill any of the roles from first in line to sitting waiting. The solution is not "labeled" with players names, and there is no way to predict who will fill each specific role. The qualitative property, the share in each role, is all we can predict.
Practice Questions

1. Downloading movies is easy, but uses a lot of bandwidth on the internet. Consider 100 students sharing a dormitory with limited connectivity to the internet. Each student can download a movie or watch television. Unlike downloading movies, one additional person watching TV does not affect the signal available to others. Assume that watching TV is valued at 25 units of utility. Assume that value of downloading movies depends on the speed of downloading, which is equal to R-N, where R is a fixed number that depends on the internet capacity in the dormitory and N is the number of students downloading.

(a) Suggest a state variable and representative agent for this game.  
(b) Find the equilibrium number of students who download movies if R = 75. What is the speed of downloading in equilibrium?  
(c) Imagine that students lobby to have the connectivity improved, and succeed in getting the capacity increased so R = 100. Does this improve the equilibrium speed of downloading?  
(d) Could a tax on downloading that was spent to improve the quality of TV make students better off?

2. Consider an object being auction to a group to two bidders. The object is worth nothing to the seller, but nobody except the bidders themselves know how much the object is worth to the bidders. Each of the following four possible combinations of bidder values is equally likely:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Combination of Values</th>
<th>Likelihood of these Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bidder 1</td>
<td>Bidder 2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Assume that the seller decides to use a second price sealed bid auction.

(a) Write down the Normal form of the second price auction.  
(b) State in words why it is in the interest of players to bid their valuation.
(c) The simple second price auction assumes the seller does not set a “reservation price” and so sells the object for sure. Do you think the seller can benefit from a reservation price? Why? Is it efficient for the seller to set a reservation price (other than zero)?
Chapter 8 Evolutionary Games

This final set of lectures brings together many of the ideas from earlier in the course, but with a new interpretation. We examine a model of evolution with random mutation and natural selection built on and reinterpreting the elements of game theory from the previous lectures. We introduce the “biological” interpretation of strategies as “behaviours” rather than choices. Many sophisticated mathematical results have been developed by evolutionary game theorists. Some of the most interesting results can be grasped with a modest amount of mathematics and a few simple diagrams, directly related to the material in previous lectures.

This opens the door to broader reinterpretations of now familiar results. The evolutionary model can be used to interpret learning. Rather than thinking of rational individuals choosing the best strategy, we think of strategies “surviving” based on some criterion of success. In some cases, this learning can lead back to Nash Equilibrium.

Key terms: population game; evolutionary stable strategy; replicator dynamic; bounded rationality

1. New Interpretations of Familiar Concepts

The simplest evolutionary model is based on a combination of “random mutation” and “natural selection”. Random mutation is the engine that causes change. Random means that change occurs without being directed toward some higher purpose. There is no reason to expect random mutations to be “improvements”; many maybe most will be “failures”. Natural selection is “survival of the fittest.” Mutations that are improvements survive to reproduce themselves. This is the process that steers evolution, controlling the cumulative direction of the changes induced by random mutation. Fitness is not an absolute concept. It is relative, and depends on the context -- or ecology -- in which the mutation occurs. A mutation that prospers in one environment may fail entirely in another. This interactivity is very much in the spirit of game theory, where strategies are only occasionally good in an absolute sense, but are instead best responses to the strategies chosen by other players.

We will focus on some basic models of evolutionary game theory that can be understood using now familiar elements of traditional rational choice game theory. There are three complementary reinterpretations:
1. We reinterpret strategies as “hard-wired behaviour”, rather than the object of choice. You might think of these as “instincts.” As long as behavior is determined for every possible situation, instincts are equivalent to a description of a single strategy. These strategies might be embodied in “phenotypes,” -- varieties of organism, mechanically restricted to perform certain behaviour-- or computer programs that are literally programmed to behave in a certain way, or even habits and belief systems that compel individuals to make certain choices, independently of their “payoff.” What these interpretations have in common is that agents do not choose a strategy, but simply behave in particular ways in given situations. A strategy is a pattern, not a plan. Remember we never require strategies to be “good” in any sense, just well-defined, and complete in their description of behaviour in all situations. The same will be true of strategies in this reinterpretation.

2. Strategies are engaged in a repeated game. We reinterpret a round of play as occurring between randomly chosen members of very large populations of strategies. Games are still (typically) played between two players, but now these players are drawn from the population and “matched”. Many of these matches are played in each round. Since populations are very large, the average payoff from a round of play between randomly matched strategies is determined by the population shares of the various strategies. From the perspective of any given strategy, it’s as if it is facing another player choosing a mixed strategy, with probabilities over the pure strategies given by the population shares of those strategies.

3. Finally, we reinterpret payoffs as “reproductive fitness”: relatively more successful strategies gain share in the population. This is natural selection. This is also the dynamic structure of evolutionary games. Evolutionary games are repeated games where pairs of players are drawn randomly to play a stage game. After each round of the stage game, payoffs are calculated and determine the population shares for the next round of the evolutionary game. Many possible “dynamics” could guide this process; we will focus on a widely used rule for selection, the replicator dynamic.

Equilibrium in an evolutionary game is a “stable set of strategies.” This is a rest point in the reproductive process, where the population shares of strategies is unchanging over time.

---

27 The stage game may itself be another repeated game, for example a 100 round repeated Prisoner’s Dilemma or a 100 move Centipede Game. Typically we will assume the stage game is a one shot game.
Mutation

The traditional game theory models we’ve studied are all based on well defined players with fixed sets of possible strategies, and have no role for random variation of strategies. With large populations of strategies, one simple way to include random variation is to change the population shares of strategies already present in the population. Another possibility is to allow new strategies to emerge. From a practical point of view, these new strategies might be ones we know about. In an evolutionary model of the 100 round repeated Prisoner’s Dilemma, for example, we can ask about how a population of tit-for-tat strategies fares when some small group of “pure defectors” is added to the mix. Another possibility is to use computer algorithms that combine and modify existing strategies to form genuine “mutants” that enter and compete against the existing population.

Whatever the source of this mutation, a strong definition of evolutionary equilibrium should require that the population not only reproduces itself, but that it can withstand the addition (invasion, infection) of small bands of mutant strategies.

2. The Selfish Herd

Before we formally develop some of the ideas of evolutionary game theory, consider an example of how a social dilemma might arise within a group of animals. Herd behavior appears to be very social, and it might be easy to assume that herd animals derive direct benefit from close contact with fellow creatures. The zoologist WD Hamilton developed a model of herding based on the idea of “cover seeking.” Hamilton argues that many prior researchers erred in assuming that herding was beneficial to the collective of animals, offering an example of cooperation for mutual gain. Instead, he argued, average fitness often declines when animals cluster into herds. For example, herds put more pressure on resources, and are more conspicuous to predators. Despite this, individual animals benefit from hiding behind other members of their species that are non-threatening and provide cover from predators. The conflict between individual and collective interest makes this a classic social dilemma. Herds are too large and

---

dense from the perspective of average welfare, but individuals do not gain from leaving the pack.

Below I discuss briefly how we can think about selection and the dynamics of evolution in a social dilemma, the prisoner’s dilemma. First, we’ll look at a game without dominant strategies, and relate the evolutionary equilibrium to mixed strategy Nash equilibrium.

3. The Hawk-Dove Game

The game is played within a populations composed of two strategies, an aggressive “Hawk” and a passive “Dove”. Interactions are assumed to be random, in that the probability that a given “bird” meets a Hawk is determined by the share of Hawks in the total population, and similarly for meeting Doves. Since there are only two types of birds in the model, it can be solved with a single state variable, which we’ll take to be the share of Hawks. Individual members of each of the bird populations are identical, so can be characterised by a representative agent.

Representative Agents: Hawk; Dove

State variable: \( p \) the share of Hawks in the population.

The payoffs from interacting with each bird type can be organized into a simple “fitness table”:

<table>
<thead>
<tr>
<th>HAWK-DOVE</th>
<th>Matches with:</th>
<th>Hawk ( p )</th>
<th>Dove ( (1-p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Hawk</strong></td>
<td></td>
<td>-25</td>
<td>14</td>
</tr>
<tr>
<td><strong>Dove</strong></td>
<td></td>
<td>-9</td>
<td>5</td>
</tr>
</tbody>
</table>

where the \( p \) and \((1 - p)\) indicate the population shares of the two strategies.
4. Equilibrium

Evolutionary equilibrium refers to stable population shares (the total population might be growing or shrinking, but the share of each strategy is constant). Population shares in one round depend on the success of the strategies in the previous round. Therefore, for population shares to be stable the reproductive success of each strategy must be equal; otherwise, one would grow relative to the other. Notice that this is exactly how we defined the conditions for a mixed strategy Nash equilibrium: each strategy must be equally successful or a “bird” would not be indifferent between “playing Hawk” and “playing Dove.” Here it is not indifference that counts, but equal fitness. The formal computations are identical even though the interpretation is very different.

To find the stable populations shares we need to find a value of $p$, the probability that the opposing bird is a Hawk, so that the expected payoff (i.e. fitness) of Doves equals that of Hawks. Recall that $p$ is the share of Hawks in the population of birds, and meetings are random. The expected utility, which I will call fitness, of a Hawk, is the probability of meeting another Hawk times the payoff from meeting another Hawk, plus the probability of meeting a Dove times the payoff from meeting a Dove:

$$Fit_H = -25p + 14(1 - p) = 14 - 39p$$

In equilibrium, this must be equal to the expected utility of a Dove:

$$Fit_D = -9p + 5(1 - p) = 5 - 14p$$

Setting these equal to each other (so reproductive success is identical) yields the equilibrium share of Hawks. For the numbers chosen here, this is $p = 0.36$.

5. Evolutionary Stable Strategies

The population share just computed is an equilibrium, in that there is no force within the model that would cause the shares to vary: each type of bird is equally successful (on average) in each round, so the population shares remain constant. However, we would like a stronger definition of equilibrium, where the composition of the strategy population is able to resist random mutation. What would happen if a “genetic” mutation occurred that transformed Doves into Hawks or vice versa. If the equilibrium is unstable, so small changes in population shares causes the shares to move further and
further from the equilibrium over time, it is very unlikely that we’d ever observe the equilibrium in the first place.

Evolutionary Stable Strategies (ESS) are strategies that are “immune” to small invasions (or infections) of other strategies. An “incumbent” strategy is an ESS if for invasion by a small population of mutant strategies. One way to define an ESS, is that is must satisfy two requirements:

**ESS requirement 1.** Any invader can do no better against the incumbent than the incumbent does against itself.

If this were not true, the invader strategy would take over and grow to dominate the population. This is sensible enough. But recall that in a mixed strategy equilibrium in which the two strategies are both played with some probability, each must do equally well as the mixed strategy itself. In particular, a mutant strategy in the hawk-dove game that didn’t play hawk with probability .36, but played hawk for sure would do as well against the equilibrium mixed population as the equilibrium population does against itself.

This suggests

**ESS requirement 2.** If the invader does equally well against the incumbent as the incumbent does against itself, then the incumbent must do strictly better against the invader than the invader does against itself.

That’s going to make it hard for a mutant to prosper. This is satisfied in Hawk-Dove as strategy Hawk does much worse against Hawk (itself) than it does against the incumbent mixed population, the ESS. (The ESS has a fitness value of 0 against a Hawk, while against itself, Hawk has a fitness value of –25.)

6. *The Replicator Dynamic*

The definition of ESS just given can be checked directly from looking at the fitness of the strategies. In the class of games we are considering, Evolutionary Stability is also equivalent to “stability” under a particular selection process, the replicator dynamic.

The replicator dynamic is a rule for the change in population shares of strategies in between rounds of an evolutionary game. It requires that the population share of a strategy increases in proportion to the difference between the fitness for that strategy
and the average fitness. Relatively successful strategies gain share in the population. It is the “selection” principle in our model of evolution. Notice that a strategy that doesn’t exist cannot be successful, so the replicator dynamic will never return a strategy from extinction. It can, on the other hand, drive a strategy into extinction.

The replicator dynamic can be expressed in a fairly simple equation. What the replicator dynamic says is that for every strategy $i$, the population share of $i$ will change according to

$$\Delta p_i = p_i(Fit_i - AverageFit)$$

The variable $p_i$ is the population share of strategy $i$. For two strategy games, like Hawk Dove, the population shares add to one, so we need only one share as a state variable. This means only one replicator dynamic equation is needed to describe the change in both population shares. The symbol $\Delta$ (pronounced delta) simply means change$^{29}$; it says how the strategy $i$ is changing over time. $Fit_i$ is the fitness (expected payoff) for strategy $i$, and $AverageFit$ is the mean fitness in the whole population.

For symmetric games of two strategies this can be written very simply. For the Hawk – Dove game we have already worked out most of what we need to write down the replicator dynamic.

The payoff values given above were:

<table>
<thead>
<tr>
<th></th>
<th>Hawk $p$</th>
<th>Dove $(1-p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hawk</td>
<td>-25</td>
<td>14</td>
</tr>
<tr>
<td>Dove</td>
<td>-9</td>
<td>5</td>
</tr>
</tbody>
</table>

Fitness for Hawk is

$$Fit_H = -25p + 14(1 - p) = 14 - 39p$$

and for Dove, it is

$$Fit_D = -9p + 5(1 - p) = 5 - 14p$$

$^{29}$ If we were to be more precise, this should be the time derivative of $p$, sometimes written as $\dot{p}$. 
All we need is the “average fitness”. The average fitness is just the expected fitness: the fitness of a randomly chosen strategy. If we were to draw one strategy at random, there is a $p$ chance it is a Hawk and a $(1 - p)$ chance it is a Dove. So on average fitness is

$$AverageFit = (1 - p)Fit_D + pFit_H$$

Notice by collecting terms this can be written as

$$AverageFit = Fit_D + p(Fit_H - Fit_D)$$

Now we can work out the Replicator Dynamic for this game of Hawk-Dove

$$Δp = p(Fit_H - Fit_D - p(Fit_H - Fit_D)) = p(1 - p)(Fit_H - Fit_D)$$

Substituting in the payoffs, this becomes

$$Δp = p(1 - p)(9 - 25p)$$

**Equilibrium**

In equilibrium the population shares remain unchanged from one “generation” to the next. Equilibrium requires the replicator dynamic equation to equal zero. Since the replicator dynamic is the product of three terms, when any one (or more) of these is zero, the whole expression will be zero.

1. **No Hawks**: when $p = 0$,
   $$Δp = 0 * (1 - 0)(9 - 25 * 0) = 0$$

   If there are no Hawks this year, there won’t be any next year, since it takes a Hawk to make a Hawk.

2. **Only Hawks**: when $p = 1$,
   $$Δp = 1 * (1 - 1)(9 - 25 * 1) = 0$$

   Similarly, if there are no Doves this year, there won’t be any next year, so the share of Hawks will remain at one.
3. A mixed population. This occurs when the third term of the expression is equal to zero. This requires that

\[ 9 - 25p = 0 \]
\[ p = \frac{9}{25} = 0.36 \]

This is the value we found before: when 36% of the birds are Hawks the growth rates of each population are the same, so the population shares are constant.

Population Change

So far we have only looked at rest points, where population shares are constant between rounds of the evolutionary game. It is also of interest to see what happens when the ratio of the two types of bird isn’t at any of these stationary population shares. Away from these rest points, the population is changing. Where does this change take the population? If the change takes the population to a rest point, then the change stops.

The replicator dynamic shows both the direction and speed of change. We will see that only some of the rest points can be reached by the replicator dynamic. When the replicator dynamic moves the population onto a rest point, we say that point is “stable under the replicator dynamic.” For games like those we are studying here, the condition for a rest point to be an ESS is that it is stable under the replicator dynamic.

To repeat, in the Hawk-Dove game, the replicator equation is

\[ \Delta p = p(1-p)(9 - 25p) \]

The replicator equation shows how the speed and direction of change depend on the size of \( p \), i.e. the existing population shares at some point in time. Since \( p \) represents a population share, we are only interested in values of \( p \) between zero and one. Everywhere except at those two extremes, i.e. the “boundary values” \( p = 0 \) and \( p = 1 \), the first two terms in the replicator dynamic are both positive. Therefore away from the boundaries, the direction of populations depends only on the third term in the product.

The share of Hawks grows when \( \Delta p > 0 \). This requires that the third term is positive:

\[ 9 - 25p > 0, \]

or
This means that when the population share of Hawks is less than 0.36, the share of Hawk will grow. With a relatively low share of Hawks, Hawks perform better than the average bird, so their population grows between generations. In contrast, $\Delta p < 0$ when the share of Hawks is greater than 0.36. Where there are a lot of Hawks about, both Hawks and Doves suffer, but the Hawks fare worse than the Doves.

All of this can be seen by on a diagram, familiar from our examination of mixed strategies earlier in the course.

The replicator dynamic is represented by the red arrows on the horizontal axis, which indexes $p$, the share of Hawks. The three circles on the horizontal axis indicate “rest points” of the replicator dynamic -- Hawk population shares that will not change under the replicator dynamic. The solid black circle is stable under the replicator dynamic: if the population share $p$ is nearby, the replicator dynamic will push it toward this point (36% Hawks).

The two endpoints are the boundaries on the feasible range of $p$, namely $p = 0$ and $p = 1$. As we saw from the algebra, these are rest points: if one strategy disappears, the replicator dynamic will not bring it back into existence. Notice however, that even though these are rest points, in the Hawk Dove game the boundaries are not stable under the replicator dynamic. If the population share of Hawks is close to one of these points, no matter how close, it will be driven further and further away by the replicator dynamic. You can think of the case where a population of all Doves ($p = 0$) is invaded by a tiny proportion on Hawks. At first, the Hawks meet almost only Doves, prosper, and
grow in population share. Eventually, as the share approaches 0.36 the two strategies converge in fitness. In the limit after many generations, the population share settles at this stable rest point. The fact that the boundaries are unstable rest points is indicated by the open circles.

The solid fitness lines in the figure illustrate the underlying pressure on the population shares. The key for the replicator dynamic is whether a strategy is more fit than average. The average is shown with the dotted line. As you would expect, the average fitness is equal to the fitness of both populations when they are equal to each other (p = 0.36). When one strategy totally disappears, the average is equal to the fitness of the remaining strategy. Away from these three points, the average lies between the two fitness lines.

As you can check from the algebra above, with only two strategies, being above average is equivalent to being better than the other strategy. This isn’t true when there are more than two strategies, where for example the “second best” strategy can also be above average, and therefore can grow under the replicator dynamic.

As mentioned above, ESS are stable under the replicator dynamic. We see that in addition to being a mixed strategy Nash equilibrium in the classic Hawk Dove game, the situation where there is a 0.36 chance to meet an opponent playing Hawk is also an ESS in the evolutionary game.

7. The Replicator Dynamic in Symmetric Two Strategy Games

Deriving the replicator dynamic equation for Hawk Dove took a little bit of algebra, but the equation itself is simple. There is a shortcut which will allow us to directly write down the replicator dynamic equation from the fitness payoffs for any (two strategy, symmetric) game. This shortcut formula allows you to easily compute the replicator dynamic for the games we’ve studied before, such as the Prisoner’s Dilemma, Stag-Hunt, the Driving Game, etc, without solving these equations each time. The shortcut requires a small transformation of the fitness table, which lets you plug the values directly into the replicator dynamic. One benefit of this derivation is it allows you to see that the total variety of situations in symmetric two strategy games is quite limited.

**The shortcut**

This short cut works because we can simplify the payoff matrix for symmetric two strategy games without affecting the replicator dynamic. We can then take the numbers directly from the fitness and put them in a simple formula for the replicator dynamic.
The modification is this: for each column, subtract the payoff on the main diagonal from all the payoffs in that column. Call the result a “reduced fitness matrix”. It will always have zeros on the main diagonal.

For the Hawk Dove we rewrite payoffs as

<table>
<thead>
<tr>
<th></th>
<th>Reduced HAWK-DOVE</th>
<th>Matches with:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Hawk p</td>
</tr>
<tr>
<td>Hawk</td>
<td></td>
<td>-25-(-25)</td>
</tr>
<tr>
<td>Dove</td>
<td></td>
<td>-9-(-25)</td>
</tr>
</tbody>
</table>

Remember that subtracting a negative number is the same as adding. So the payoffs are now

<table>
<thead>
<tr>
<th></th>
<th>Reduced HAWK-DOVE</th>
<th>Matches with:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Hawk p</td>
</tr>
<tr>
<td>Hawk</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Dove</td>
<td></td>
<td>16</td>
</tr>
</tbody>
</table>

First let’s work out the replicator dynamic for the reduced fitness table. The zeros make this easy. The basic components are $\text{Fit}_H = 9(1 - p)$ and $\text{Fit}_D = 16p$. Putting this into the replicator equation

$$\Delta p = p(1 - p)(\text{Fit}_H - \text{Fit}_D)$$

and gathering together the terms multiplied by $p$ yields:
Δp = p(1 − p)(9 − (9 + 16)p) = p(1 − p)(9 − 25p)

Comparing this to what we found above for the non-reduced fitness table in the Hawk Dove game confirms that reducing the fitness table does not change the replicator equation.

A General Reduced Fitness Table

For a general game, we will label the strategies $S_1$ and $S_2$, and let $p$ represent the population share of $S_1$. Once you subtract the payoff on the main diagonal from the payoffs in its column you always get something that looks like:

<table>
<thead>
<tr>
<th>General Reduced Fitness Value Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>$S_1$</td>
</tr>
<tr>
<td>$b$</td>
</tr>
<tr>
<td>$S_2$</td>
</tr>
</tbody>
</table>

Of course, different games give you different values for $a$ and $b$, but this is the only difference. Any two games with the same values for these numbers in the simplified fitness table have the same replicator dynamics, and so the same stable rest points and ESSs.

Using the pattern from the Hawk-Dove game above (where $a = 9$ and $b = 16$), you can see that the replicator dynamic for this general case is

Δ$p = p(1 − p)(a − (a + b)p)$

With this formula, several things can be directly demonstrated about the replicator dynamic in all 2X2 symmetric games. These generalize the results from the Hawk-Dove game:

- Δ$p = 0$ whenever either $p = 0$ or $p = 1$. The boundary points are always rest points.
• When $p$ is greater than zero and less than one, the first two terms are both positive. They affect the speed but not the direction of change. For these “interior” values of $p$, the direction of change depends only on the third term. Four possibilities exist: the third term is zero for every value of $p$ between zero and one; it is positive for every value of $p$ between zero and one; it is negative for every value of $p$ between zero and one; or it changes sign once at some value $p$ between zero and one. Since the sign change is either from positive to negative or vice versa, there are in total five cases to consider.

• If $a = b = 0$, then $\Delta p = 0$ at every $p$ between zero and one, and so every population share will remain constant under the replicator dynamic. In this “trivial” game, all strategies are equally fit under for any population shares, and every population mix is an ESS. This isn’t surprising when you think about the reduction of the fitness table: $a = b = 0$ when both strategies $S_1$ and $S_2$ do equally well against themselves and each other. Clearly in this case, “natural selection” has no force. Random mutation shifts the population to a new rest point; population shares may “drift” over time, but fitness plays no role.

• When $a - (a + b)p > 0$ the share of $S_1$ grows. If this is true for every value of $p$ between zero and one, there can be no interior rest point, and the population is driven toward the boundary where $p = 1$. $S_2$ is driven out by the replicator dynamic. This can be true for many different values for $a$ and $b$ (e.g. $a = 0$ and $b = -1$). In this case, the boundary point $p = 1$ is “stable under the replicator dynamic.”

• Conversely, when $a - (a + b)p < 0$ the share of $S_1$ falls. If this is true for every value of $p$ between zero and one, then $S_1$ is driven out by the replicator dynamic, $p$ goes to zero and the population becomes dominated by $S_2$. In this case, the boundary point $p = 0$ is “stable under the replicator dynamic.”

• The fourth possibility is that the values of $a$ and $b$ are such that the sign of this third term in the replicator dynamic equation depends on $p$, and the equation switches sign at some value of $p$ between zero and one. At this point of transition, the third term will equal zero:

\[ a - (a + b)p = 0 \]
\[ \text{Then} \]
\[ p^* = \frac{a}{a + b} \]
Clearly, since \( p \) is zero and one, this can only be true for certain values of \( a \) and \( b \). For example, if \( a \) is positive and \( b \) is negative, then \( p^* \) will never be between zero and one. In fact, you can convince yourself that both \( a \) and \( b \) must have the same sign or there be no interior rest point for the replicator dynamic. This fourth case has two sub-cases depending on the sign of the replicator equation to the left and right of \( p^* \).

- Since the replicator dynamic equation changes sign at \( p^* \) if \( \Delta p > 0 \) when \( p < p^* \) then it must be that \( \Delta p < 0 \) when \( p > p^* \). This means when the share \( S_1 \) is below \( p^* \) the replicator dynamic causes it to rise, while when it is above \( p^* \) the replicator dynamic causes it to fall. Therefore starting at any initial shares, the population is driven to the rest point of \( p = p^* \), which is “stable under the replicator dynamic”. It is therefore an ESS. When this interior rest point is stable, the boundary points must not be stable.

- Conversely, in the other sub-case, the replicator dynamic causes the population to rise when the share of \( S_1 \) is above \( p^* \) and fall when it is below \( p^* \). So \( p^* \) is a rest point, but it is “unstable under the replicator dynamic”. If the population happens to be such that exactly a \( p^* \) share of the population is of type \( S_1 \), the replicator dynamic won’t change it; but any tiny “random mutation” that moves it away from this point leads to more change and eventually the population converges to one of the endpoints, with only one strategy. Both of the boundaries will be “stable under the replicator dynamic.” You can see that this stability is “local”, not “global” a sufficiently large “mutation” that causes the population share to move across the threshold defined by \( p^* \) will move the equilibrium to the other stable boundary point.

8. Some Familiar Examples

With this short cut we can easily example the payoff matrices of games we have already encountered. This will clarify the general possibilities just discussed. An exercise asks you to work out some other cases.

**Prisoner’s Dilemma: the dynamics of a selfish herd.**

The reduced fitness matrix for the Prisoner’s Dilemma is found by the transformation
Reduced Prisoner’s Dilemma Matrix

\[
\begin{array}{c|cc}
\text{Cooperate} & p & (1 - p) \\
\hline
\text{Cooperate} & 2 - 2 & 0 - 1 \\
\text{Defect} & 3 - 2 & 1 - 1 \\
\end{array}
\]

Using the shortcut formula, we see that the replicator dynamic equation for the Prisoner’s Dilemma is

\[\Delta p = p(1 - p)(-1 - (1 - 1)p) = p(1 - p)(-1)\]

As always this is zero at the boundary points, \( p = 0 \) and \( p = 1 \). But for all other values of \( p \) it is negative. This means that Cooperation is driven out by the replicator dynamic. “Cooperators” are exploited in every encounter with a “Defector.” Even though two Cooperators do well when they meet, the Defectors who meet a Cooperator do even better. When Defectors meet they do poorly, but not so poorly as a Cooperator who meets a Defector\(^{30}\). Therefore, from generation to generation the share of Cooperators falls, until eventually the whole population is made up of Defectors.

The dominant strategy Nash equilibrium in the classic Prisoner’s Dilemma is also an ESS. Imagine that we start with a population of all Cooperators. Now, for some reason, a tiny fraction of those strategies “mutates” to become Defectors. No matter how small this share is, it will do well against the Cooperators and start to grow. Eventually, only

\(^{30}\) Note here I am referring to the untransformed payoffs. In the transformed game, cooperators matched with cooperators do equally well as defectors matched with defectors. But that doesn’t change the underlying replicator dynamic.
Defectors will exist. On the other hand, a population of all Defectors is immune to an “infection” of Cooperators. These mutant Cooperators will be outdone by the incumbent Defectors, and eventually wiped out. (This is an example of case three above, where \( a - (a + b)p < 0 \).)

Sadly, the stable population of all Defectors does uniformly more poorly than did the original population of Cooperators. This story captures the essence of the selfish herd.

**The Stag Hunt**

The simplified fitness table for the Stag Hunt is

<table>
<thead>
<tr>
<th></th>
<th>Stag</th>
<th>Hare</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stag</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>Hare</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

So the formula says

\[ \Delta p = p(1-p)(-1+2p) \]

For this game, there is an interior rest point for the replicator dynamic: \( \Delta p = 0 \) when

\[ -1 + 2p = 0 \]

\[ p = \frac{1}{2} \]

When \( p < \frac{1}{2} \) there are more Hare hunters than Stag hunters in the population, and the sign of the replicator dynamic equation is negative (i.e. for \( p < \frac{1}{2}, -1 + 2p < 0 \)). This means that the population of Stag hunters declines toward zero, and eventually all that is left are Hare hunters. When there are more Stag hunters than Hare hunters the opposite is true, and eventually the whole population will hunt Stag. In this game, the two boundary points that are stable, and the mixed population equilibrium is unstable. If exactly half the hunters use each strategy, this division will persist over time; but any shift, no matter how tiny, will set off a graduate process of change that leads one of the strategies to extinction.
How the replicator dynamic acts on the possible values of $p$ is illustrated in the following diagram:

![Diagram showing the replicator dynamic](image)

Again, the red arrows indicate the direction of change under the replicator dynamic, the solid circles are stable rest points and the empty circle is an unstable rest point.

Here you can see that ESS is not equivalent to mixed strategy Nash Equilibrium. In the classic Stag Hunt, both players “mixing” and playing Stag with probability 0.5 is a mixed strategy NE. But it is not stable under the replicator dynamic, and is not an ESS. An small mutation that turns some of the Stag hunters into Hare hunters (or vice versa) will lead to an inexorable movement to one of the stable boundary equilibria. (This is an example of case four, sub-case two from above, where the sign of $a - (a + b)p$ switches from negative to positive as $p$ increases.)

### 10. Non-linear fitness values

Despite the large population of strategies, in the games above all the fitness values are determined by pair-wise play. Once two players are matched in a given round, there is no role for “externalities” due to the play of other pairs. This contrasts with games we saw in the previous lecture, like the Commuter Game, where many players played at once. The result of this pair-wise play is that the fitness is linear in the population shares (i.e. the fitness function in the figure above are straight lines). This obviously limits the number of intersections between two fitness lines to being zero, one, or an infinite number (when $a = b = 0$). This allows the simple taxonomy of solutions based on those two reduced fitness table numbers.

More generally, fitness can depend directly on the population shares of strategies. For example, it may be that as there are more Hawks in a given region, the payoff for any two Hawks meeting falls. Perhaps instinct causes them to become more aggressive to other of their own kind as populations increase. Then the fitness values in the tables need to account for $p$. This can lead to non-linear fitness functions, with more intersections and more rest points under the replicator dynamic. These rest points will generally alternate between stable and unstable.
11. Altruism

Through most of the course we have assumed that payoffs are based on “revealed preferences.” These are what they are: if players care about each other and prefer to behave “altruistically” sacrificing their own narrow interest to benefit the wider community, so be it. Of course, it remains of interest to see if self interested players can sustain cooperation to serve the collective interest. And we did see that rational players might, under certain circumstances, use promises and threats to enforce cooperation.

In evolutionary game theory, reproductive success depends not on some group benefit, but instead on how individual strategies fare in competition with each other. So-called reciprocal altruism refers to the type of cooperation we saw in traditional game theory, where, for example, strategies encoded with the instinct to cooperate conditionally but punish defection can emerge as winners in selection contests. These issues have been explored extensively using computer simulations, in which randomly matched strategies compete in tournaments consisting of many rounds, where the stage game is the indefinitely repeated prisoner’s dilemma.

Kin Selection

“Genuine altruism” where an individual can sacrifice its own reproductive fitness in favour of a larger group can be traced to relationships between “instances” of a strategy. For example, it is not surprising that an animal might evolve to protect the interests of its offspring, since these are direct carriers of its genes. Similarly, siblings, cousins, and other relatives have reasons to ensure that each other survive.
Other theories exist as to why individuals might be selected to favour genuinely altruistic traits. This is an active area of research in evolutionary biology, economics, sociology, and ethics, among other disciplines.

12. Learning

For the most part in this class we’ve assumed that players are rational, and always choose best responses. In this lecture, we assumed that players just “behave” according to a strategy, and natural selection sorts out the equilibrium. We can take these evolutionary arguments back into the world of human players by relaxing the assumption that players can always work out exactly what the best choice is. Instead, we assume people are **boundedly rational**: they intend to make best choices, but are not quite capable of doing so. Perhaps the world is too complex, or maybe they could work it out, but it takes time, and they have other things to do.

With this in mind, we can think of our players experimenting or perhaps imitating others. This leads to “mutations”. Then if players can imitate the better strategies, more successful strategies will grow in share as they “survive in competition” with less successful strategies. Better strategies are more likely to be adopted and retained. Depending on how exactly this process works, the dynamics of learning can become as described by the replicator dynamic. Thus learning may lead to an ESS, which is then also a Nash equilibrium. By imitating more successful co-player, even boundedly rational players can learn to play Nash equilibria.

As we’ve seen, not every Nash equilibria will be found by the replicator dynamic. So boundedly rational learning is yet another “refinement” to the set of Nash equilibria.

---

**Practice Questions:**

1. Find the replicator dynamic equation for Matching Pennies and Chicken. How many stable rest points are there in these games? Compare these with the Nash equilibria for these games in traditional rational choice game theory.
2. Driving takes coordination, not only for the person doing it, but also among the various drivers on the road. One important issue is which side of the road people should drive on. Imagine that there are two towns located on either side of a long hill. There are a large number of drivers making the trip from time to time. Traffic between the towns is sparse, but not infrequent.

Consider two drivers headed in opposite directions between the towns. As they approach the top of the hill, each can choose to drive on the left or right. If both choose either left of right, they pass safely on opposite sides of the road. If one chooses to drive on her left while other drives on his right, they collide.

Assume that the drivers have symmetric preferences, as in the following diagram

<table>
<thead>
<tr>
<th></th>
<th>Driver 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Driver 1</td>
<td>10,10</td>
</tr>
<tr>
<td>Left</td>
<td>-10,-10</td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
</tbody>
</table>

First assume that these players are fully rational and choose strategies to maximize their expected utility.

(a) Compute all of the Nash Equilibria for this game.

Now assume that players are boundedly rational. Drivers want to get over the hill safely, but don’t necessarily choose the optimal strategy. Since there are a large number of trips made by different drivers, one never knows who will be coming over the top of the hill.

Assume that most of the time drivers just choose a side from habit. But occasionally, before leaving on a trip across the hill, some small proportion of the drivers consult recent accident reports, and learn about the latest crashes. When one side seems safer, these drivers switch sides if necessary to use the safer side.

(b) Is the process by which drivers choose strategies described in the previous paragraph going to lead to the replicator dynamic? Explain.
(c) Use the numbers in the Normal Form above to write down the Fitness table for this game. Find the simplified fitness table.

(d) Find all of the rest points of the replicator dynamic.

(e) Which, if any, of the rest points are ESS?

(f) What can you say about the drivers’ likelihood of learning to play the Nash Equilibria of this game?