MILNOR-WITT K-GROUPS OF LOCAL RINGS

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Abstract. We introduce Milnor-Witt K-groups of local rings and show that the nth Milnor-Witt K-group of a local ring R which contains an infinite field of characteristic not 2 is the pull-back of the nth power of the fundamental ideal in the Witt ring of R and the nth Milnor K-group of R over the nth Milnor K-group of R modulo 2. This generalizes the work of Morel-Hopkins on Milnor-Witt K-groups of a field.

0. Introduction

The Milnor-Witt K-theory of a field R, denoted $\mathrm{K}_*^{MW}(R) = \bigoplus_{n\in\mathbb{Z}} \mathrm{K}_n^{MW}(R)$, arises as an object of fundamental interest in motivic homotopy theory; namely, as the "0-line" part of the stable homotopy ring of the motivic sphere spectrum over R, see Morel [19]. Beginning from an initial presentation discovered in collaboration with Hopkins, Morel [18] showed that, for a field R, the group $\mathrm{K}_n^{MW}(R)$ is the pull-back of the diagram

$$\begin{tikzcd} & \mathrm{K}_n^M(R) \arrow[d, "e_n"] \\ \mathrm{I}^n(R) \arrow[r] & \mathrm{I}^n(R)/\,\mathrm{I}^{n+1}(R)\,, \end{tikzcd} \tag{1}$$

where $\mathrm{K}_n^M(R)$ denotes the nth Milnor K-group of R, the symbol $\mathrm{I}^n(R)$ the nth power of fundamental ideal $\mathrm{I}(R)$ in the Witt ring $\mathrm{W}(R)$, and e_n maps the symbol $\ell(a_1)\cdot\ldots\cdot\ell(a_n)$ to the class of the Pfister form $\ll a_1,\ldots,a_n \gg$. (Here $\mathrm{I}^n(R) = \mathrm{W}(R)$ is understood for $n<0$.) In this form, Milnor-Witt K-groups of fields already appeared implicitly in earlier work of Barge and Morel [8], who used them to introduce oriented Chow groups of an algebraic variety, a theory later elaborated in the work of Fasel [11, 12]. This so-called Chow-Witt theory was used by Barge and Morel [8, 7] to construct an Euler class invariant for algebraic oriented vector bundles which, in analogy with its topological counterpart, is the primary obstruction to the existence of a nowhere-vanishing section, see Morel [20, Chap. 8]. This circle of ideas has since been greatly elaborated upon, see for instance the article [1] by Asok and Fasel for some very recent developments.

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A further area where Milnor-Witt K-groups of a field show up is the theory of framed motives, see for instance the recent work of Neshitov [21].

We introduce in this work Milnor-Witt K-groups $\mathrm{K}_*^{MW}(R)$ of a local ring R. Our definition is the naive generalization of the Morel-Hopkins presentation given in Morel's book [20, Def. 3.1], or Morel [18, Def. 5.1], *i.e.* $\mathrm{K}_*^{MW}(R)$ is a $\mathbb{Z}$-graded $\mathbb{Z}$-algebra generated by an element $\hat{\eta}$ in degree -1 and elements $\{a\}$ (a a unit in R) in degree 1 modulo four relations, see Definition 5.1. This definition has also been considered in the recent work of Schlichting [26], where (extending earlier work of Barge and Morel [7] and Hutchinson and Tao [14] over fields), it is shown that Milnor-Witt K-groups arise as obstructions to integral homology stability for special linear groups over local rings with infinite residue fields. This in turn is used then by Schlichting [26, §6] to extend results of Morel on the vanishing of Euler class for oriented vector bundles over affine schemes.

Our main result about these groups is the following, see Theorem 5.4.

Theorem. *Let R be a local ring which contains an infinite field of characteristic $\neq$ 2. Then the nth Milnor-Witt K-group $\mathrm{K}_n^{MW}(R)$ is the pull-back of the diagram (1) for all $n \in \mathbb{Z}$.*

The proof uses a presentation of $\mathrm{I}^n(R)$ given in [13], whose proof is based on a recent deep theorem of Panin and Pimenov [24] on the existence of unimodular isotropic vectors for quadratic forms over regular local rings.

Another ingredient of our proof which is of some interest on its own is the following. In Section 2 we show that over an arbitrary local ring, which contains $\frac{1}{2}$, two Pfister forms are isometric if and only if they are chain p-equivalent, see 2.1 for the definition. This has been shown for fields by Elman and Lam [10] more than 40 years ago and seems to be new for local rings (although most likely known to experts).

In the last section we show for a regular local ring R containing an infinite field of characteristic not 2 with quotient field K that the kernel of

$$\mathrm{K}_n^{MW}(K) \xrightarrow{(\partial_P)_{\mathrm{ht}\, P=1}} \bigoplus_{\mathrm{ht}\, P=1} \mathrm{K}_{n-1}^{MW}(R_P/PR_P)\,,$$

where ∂_P are the residue maps introduced by Morel [20, Sect. 3.2], is naturally isomorphic to $\mathrm{K}_n^{MW}(R)$ for all $n \in \mathbb{Z}$. This implies via an argument of Colliot-Thélène [9, Sect. 2] that the unramified Milnor-Witt K-groups are birational invariants of smooth and proper schemes over an infinite field of characteristic not 2.

We want to point out that our proof of the main theorem does not use results from Morel's work [18], although we have certainly borrowed ideas from there. Actually we prove the theorem simultaneously also for fields of characteristic not 2.

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1. Quadratic forms over local rings

1.1. We recall in this section some definitions and results of the algebraic theory of quadratic forms over local rings. We refer for proofs and more information to Scharlau [25, Chap. I, §6].

We start with the following important

Convention. *Throughout this work we assume that all rings are commutative with* 1 *and contain* $\frac{1}{2}$. *In particular, we assume that fields are of characteristic not* 2.

1.2. *Definitions.* Let R be a local ring. A *quadratic form* over R is a map $q : V \longrightarrow R$, where V is a free R-module of finite rank, such that $q(\lambda v) = \lambda^2 q(v)$ for all $\lambda \in R$ and $v \in V$, and $b_q(v,w) := q(v+w) - q(v) - q(w)$ is a symmetric bilinear form on V. Throughout this work we assume as part of the definition that quadratic forms are non singular (also called regular), *i.e.* the associated bilinear form b_q is not degenerate. The pair (V,q) is called a quadratic space.

We denote for a quadratic space (V,q) the set of unit values of q by $\mathrm{D}(q)^\times$, *i.e.* $\mathrm{D}(q)^\times := \big\{q(v)|v \in V\big\} \cap R^\times$, where $R^\times$ is the multiplicative group of units of R.

Two quadratic spaces (V_1,q_1) and (V_2,q_2) are called *isomorphic* or *isometric* if there exists an R-linear isomorphism $f : V_1 \longrightarrow V_2$, such that $q_2(f(v)) = q_1(v)$ for all $v \in V_1$. Such a map f is called an isometry, and we use the notation $(V_1,q_1) \simeq (V_2,q_2)$, or more briefly $q_1 \simeq q_2$, to indicate that (V_1,q_1) and (V_2,q_2) are isometric. The group of all automorphisms of the quadratic space (V,q) is denoted by $\mathrm{O}(V,q)$, or $\mathrm{O}(q)$ only, and called the *orthogonal group* of (V,q) respectively of q.

As usual we denote the *orthogonal sum* and the *tensor product* of two quadratic spaces (V_1,q_1) and (V_2,q_2) by $(V_1,q_1) \perp (V_2,q_2)$ and $(V_1,q_1) \otimes (V_2,q_2)$, respectively, or more briefly by $q_1 \perp q_2$ and $q_1 \otimes q_2$ only. Note that the tensor product is defined since we assume that $\frac{1}{2} \in R$.

Since $\frac{1}{2} \in R$ every quadratic space has an orthogonal basis and is therefore isomorphic to a diagonal form $< a_1, \ldots, a_n >$ for appropriate $a_i \in R^\times$. Another important consequence of our assumption that 2 is invertible in R is *Witt cancellation*, *i.e.* if $q \perp q_1 \simeq q \perp q_2$ for quadratic forms q, q_1, q_2 over R then $q_1 \simeq q_2$.

1.3. *Isotropic vectors.* Let (V,q) be a quadratic space over the local ring R. A vector $v \in V \setminus \{0\}$, such that $q(v) = 0$, is called an *isotropic*- respectively if v is moreover unimodular a *strictly isotropic vector.* If v is strictly isotropic then there exists another isotropic vector $w \in V$, such that $b_q(v,w) = 1$, *i.e.* v, w are a *hyperbolic pair* and $q|_{Rv \oplus Rw} \simeq < 1, -1 >$ is isometric to the *hyperbolic plane* $\mathbb{H}$. Then $(V,q) \simeq (V_1, q|_{V_1}) \perp \mathbb{H}$ for some subspace $V_1 \subseteq V$.

The quadratic space (V,q) respectively the quadratic form q is called *isotropic* if there exists a strictly isotropic vector (for q) in V. Otherwise (V,q) respectively q is called *anisotropic.*

1.4. *Grothendieck-Witt groups.* Let R be a local ring. The Grothendieck group of isomorphism classes of quadratic spaces over R with the orthogonal sum as addition is called the *Grothendieck-Witt group* or *ring* of R. It is in fact a commutative ring, where the multiplication is induced by the tensor product. We denote this ring by $\mathrm{GW}(R)$. The quotient of $\mathrm{GW}(R)$ by the ideal generated by the hyperbolic plane $< 1, -1 >$ is the *Witt group* or *Witt ring* $\mathrm{W}(R)$ of R. (Note that since $\frac{1}{2} \in R$ every hyperbolic space over R is an orthogonal sum of hyperbolic planes.)

1.5. *The fundamental ideal of the Witt ring.* Let R be a local ring. The *fundamental ideal* $\mathrm{I}(R)$ of the Witt ring of R is the ideal consisting of the classes of even dimensional forms. It is additively generated by the classes of 1-Pfister forms $\ll a \gg := < 1, -a >$. We denote its powers by $\mathrm{I}^n(R)$, $n \in \mathbb{Z}$, where $\mathrm{I}^n(R) = \mathrm{W}(R)$ for $n \leq 0$ is understood.

Obviously $\mathrm{I}^n(R)$ is additively generated by n-Pfister forms

$$\ll a_1, \ldots, a_n \gg := \bigotimes_{i=1}^{n} \ll a_i \gg ,$$

$a_1, \ldots, a_n \in R^\times$, for all $n \geq 1$. Note that a Pfister form q has an orthogonal decomposition $q = < 1 > \perp q'$. By Witt cancellation the form q' is unique up to isometry and called the *pure subform* of the Pfister form q. We will use throughout the notation q' for the pure subform of a Pfister form q.

For later use we recall the following identities of Pfister forms.

1.6. Examples. Let R be a local ring.

(i) Let $q = \ll a, b \gg$ be a 2-Pfister form over R. If $-c \in D(q')^\times$ then we have $q \simeq \ll c, d \gg$ for some $d \in R^\times$. This follows by comparing determinants.

(ii) Assume that the two dimensional quadratic space $\ll a \gg = < 1, -a >$ represents $c \in R^\times$. Then $< -b, ab >$ represents $-bc$ for all $b \in R^\times$. Comparison of determinants then implies $< -b, ab > \simeq < -bc, abc >$, and so we have then $\ll a, b \gg \simeq \ll a, bc \gg$.

1.7. *A pull-back diagram.* Another way to define the fundamental ideal is as follows. The *rank* of a quadratic space (V, q) is by definition the rank of the underlying free R-module V. As the rank is additive on orthogonal sums it induces a homomorphism $\mathrm{rk} : \mathrm{GW}(R) \longrightarrow \mathbb{Z}$. Since hyperbolic spaces have even rank this function induces in turn a homomorphism $\bar{\mathrm{rk}} : \mathrm{W}(R) \longrightarrow \mathbb{Z}/2\mathbb{Z}$. The fundamental ideal $\mathrm{I}(R)$ is then the kernel of $\bar{\mathrm{rk}}$.

As Witt cancellation holds for quadratic forms over R every quadratic space (V, q) decomposes up to isometry uniquely $(V, q) \simeq (W, \varphi) \perp (U, \phi)$ with φ anisotropic and (U, ϕ) hyperbolic. This implies the following well known fact.

Lemma. *Let R be a local ring. Then the diagram*

$$\begin{tikzcd} \mathrm{GW}(R) \arrow[r, "\mathrm{rk}"] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathrm{W}(R) \arrow[r, "\bar{\mathrm{rk}}"'] & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

is a pull-back diagram.

1.8. *Reflections.* A vector $0 \neq v \in V$ which is not isotropic is called *anisotropic*, respectively *strictly anisotropic* if $q(v)$ is a unit in R. A strictly anisotropic vector has to be unimodular. If $v \in V$ is a strictly anisotropic vector for the quadratic form q then

$$\tau_v \;:\; V \longrightarrow V \,,\; x \longmapsto x - \frac{b_q(v, x)}{q(v)} \cdot v$$

is a well defined R-linear map, called the *reflection* associated with the vector v. Recall that we have $q(\tau_v(x)) = q(x)$ for all $x \in V$, *i.e.* τ_v is an isometry and so defines an element of $\mathrm{O}(q)$.

1.9. *Reduction.* We continue with the notation of the last section. Given a quadratic space (V, q) over R then $(\bar{V}, \bar{q}) := k \otimes_R (V, q)$ is a quadratic space over the residue field k of R, the *canonical reduction* of (V, q). Note that if (V, q) is isotropic then also the reduction $(\bar{V}, \bar{q})$ is isotropic.

If $f : \phi \simeq \psi$ is an isometry then $\bar{f} := \mathrm{id}_k \otimes f$ is an isometry between $\bar{\phi}$ and $\bar{\psi}$. In particular, we have then a homomorphism of orthogonal groups

$$\rho_\phi \,:\, \mathrm{O}(\phi) \longrightarrow \mathrm{O}(\bar{\phi})\,,\ \alpha \longmapsto \bar{\alpha}\,.$$

This map is surjective. In fact, by the Cartan-Dieudonné Theorem, see *e.g.* [25, Chap. 1, Thm. 5.4], the group $\mathrm{O}(\bar{q})$ is generated by reflections $\tau_{\bar{v}}$ with $\bar{v} \in \bar{V}$ an anisotropic vector. If $v \in V$ is a vector which maps to $\bar{v}$ under the quotient map $V \longrightarrow \bar{V}$ then v is a strictly anisotropic vector in V, and therefore the reflection $\tau_v \in \mathrm{O}(q)$ exists. We have $\bar{\tau}_v = \rho_\phi(\tau_v) = \tau_{\bar{v}}$ which proves the claim.

2. Chain p-equivalent and isometric quadratic forms over local rings

2.1. *Definition of chain p-equivalence.* Chain p-equivalence of quadratic forms over fields has been introduced by Elman and Lam [10]. The definition carries over to rings word by word.

Definition. Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Let $n \geq 2$ be an integer, and $\phi = \ll a_1, \ldots, a_n \gg$ and $\varphi = \ll b_1, \ldots, b_n \gg$ two n-Pfister forms.

(i) The n-Pfister forms ϕ and φ are called *simply p-equivalent* if there exist indices $1 \leq i < j \leq n$, such that $\ll a_i, a_j \gg \simeq \ll b_i, b_j \gg$ and $a_l = b_l$ for all $l \neq i, j$.

(ii) The n-Pfister forms ϕ and φ are called *chain p-equivalent* if there exists a chain $\phi = \mu_0, \mu_1, \ldots, \mu_r = \varphi$ of n-Pfister forms over R, such that μ_i is simply p-equivalent to μ_{i+1} for all $0 \leq i \leq r-1$.

Following Elman and Lam [10] we use the notation $\phi \approx \varphi$ to indicate that ϕ and φ are chain p-equivalent.

The aim of this section is to show that over a local ring R, where 2 is a unit, Pfister forms are isometric if and only if they are chain p-equivalent. In the case where R is a field of characteristic not 2, this was first proved by Elman and Lam [10]. To treat the more general case, we follow essentially their arguments, but some modifications are necessary, mainly for the reason that a non-zero element in a general local ring need not to be a unit. We overcome these obstructions using Lemma 2.2 respectively Lemma 2.3 if the residue field is small.

2.2. Lemma. *Let R be a local ring whose residue field k is not the field with* 3 *or* 5 *elements. Let (V, ϕ) and (W, φ) be quadratic spaces over R, and set $q := \phi \perp \varphi$. Then given $a \in \mathrm{D}(q)^\times$ there exists $v \in V$ and $w \in W$, such that*

(a) *$a = q(v, w) = \phi(v) + \varphi(w)$, and*

(b) *both $\phi(v)$ and $\varphi(w)$ are units in R.*

Proof. Let $x = (v', w')$. We assume first that $R = k$ is a field.

If $\phi(v') \neq 0 \neq \varphi(w')$ there is nothing to prove, so assume one of these values is zero, say $\varphi(w') = 0$. Then $a = \phi(v') = q(v')$, and so we can assume that $x = v'$.

The quadratic form φ is non singular and so there exists $z \in W$, such that $\varphi(z) \neq 0$. As $R = k$ has at least 7 elements there exists $\lambda_0 \in k^\times$, such that

$$\lambda_0^2 \neq \pm \frac{\phi(v')}{\varphi(z)} .$$

Consider the vector $u := v' + \lambda_0 \cdot z$. We have $q(u) = \phi(v') + \lambda_0^2 \cdot \varphi(z)$ which is not 0 by our choice of λ_0, *i.e.* u is an anisotropic vector and so the reflection $\tau_u \in \mathrm{O}(q)$ is defined. Set now

$$(v, w) := \tau_u(v') .$$

As τ_u is in $\mathrm{O}(q)$ we have $a = q(v') = q(\tau_u(v')) = q(v, w) = \phi(v) + \varphi(w)$, and so (a). We are left to show that $\phi(v)$ and $\varphi(w)$ are both non zero. For this we compute

$$(v, w) = \tau_u(v') = \left(1 - \frac{2\phi(v')}{\phi(v') + \lambda_0^2\varphi(z)}\right) \cdot v' + \frac{2\lambda_0\phi(v')}{\phi(v') + \lambda_0^2\varphi(z)} \cdot z ,$$

and so $v = \left(1 - \frac{2\phi(v')}{\phi(v')+\lambda_0^2\varphi(z)}\right) \cdot v' =: c \cdot v'$ and $w = \frac{2\lambda_0\phi(v')}{\phi(v')+\lambda_0^2\varphi(z)} \cdot z =: d \cdot z$. By our choice of λ_0 both coefficients c, d are non zero and so

$$\phi(v) = c^2 \cdot \phi(v') \neq 0 \neq d^2\varphi(z) = \varphi(w)$$

as desired.

We come now to the general case, *i.e.* R is a local ring whose residue field k has at least 7 elements. If U is an R-module we denote by $\bar{u}$ the image of $u \in U$ in $\bar{U} = k \otimes_R U$.

Since $a \in R^\times$ its residue $\bar{a}$ in k is non zero. By the field case there are then $\bar{v} \in \bar{V}$ and $\bar{w} \in \bar{W}$, such that $\bar{a} = \bar{q}(\bar{v}, \bar{w}) = \bar{\phi}(\bar{v}) + \bar{\varphi}(\bar{w})$, and

$$\bar{\phi}(\bar{v}) \neq 0 \neq \bar{\varphi}(\bar{w}) .$$

Since $0 \neq \bar{a} = \bar{q}(\bar{v}, \bar{w}) = \bar{q}(\bar{v}', \bar{w}')$ there exists $\bar{\tau} \in \mathrm{O}(\bar{q})$, such that

$$\bar{\tau}(\bar{v}', \bar{w}') = (\bar{v}, \bar{w}) .$$

As seen in 1.9 there exists $\tau \in \mathrm{O}(q)$ whose canonical reduction is $\bar{\tau}$. We set then $(v, w) := \tau(v', w')$. Clearly we have then $a = q(v, w)$, and $\phi(v)$ and $\varphi(w)$ are both units in R since their reductions $\overline{\phi(v)} = \bar{\phi}(\bar{v})$ and $\overline{\varphi(w)} = \bar{\varphi}(\bar{w})$ are both non zero in the residue field k. □

Remark. The form $q =< 1 > \perp < 1 >$ and $a = 1$ shows that the lemma does not hold for R a field with 3 or 5 elements.

To handle also the case of local rings whose residue fields have only 3 or 5 elements we prove a more specialized lemma.

2.3. Lemma. *Let R be a local ring whose residue field k has 3 or 5 elements, and (W, φ) a quadratic space of rank ≥ 3 over R.*

(i) *Let $q = bt^2 + \varphi$ with $b \in R^\times$, and $a \in \mathrm{D}(q)^\times$. Then there is $s \in R$ and $w \in W$, such that $\varphi(w) \in R^\times$ and $a = bs^2 + \varphi(w)$ (note that we do not claim that s is a unit in R).*

(ii) *Let (V, ϕ) be another quadratic space of rank ≥ 3 over R and $q = \phi \perp \varphi$. Let $a \in \mathrm{D}(q)^{\times}$. Then there are vectors $v \in V$ and $w \in W$, such that both $\phi(v)$ and $\varphi(w)$ are units in R and $a = \phi(v) + \varphi(w)$.*

Proof. The proof uses the fact that over a finite field of characteristic not 2 every quadratic form of rank ≥ 3 is isotropic, see *e.g.* [25, Chap. 2, Thm. 3.8], and so is universal, *i.e.* represents every element of the field.

For (i) let $t_0 \in R$ and $w_0 \in V$, such that $a = bt_0^2 + \varphi(w_0)$. Then $0 \neq \bar{a} = \bar{b}\bar{t_0} + \bar{\varphi}(\bar{w}_0)$. Since $\dim \bar{\varphi} \geq 3$ there exists by the remark above $\bar{w} \in \bar{W}$, such that $\bar{\varphi}(\bar{w}) = \bar{a}$. Then there is $\bar{\tau} \in \mathrm{O}(\bar{q})$, such that $\tau(\bar{t_0}, \bar{w}_0) = (0, \bar{w})$. Let $\tau \in \mathrm{O}(q)$ be a preimage of $\bar{\tau}$ under the reduction map $\mathrm{O}(q) \longrightarrow \mathrm{O}(\bar{q})$, see 1.9. The vector $(s, w) := \tau(t_0, w_0)$ does the job.

For (ii) we use a similar reasoning. There are $v_0 \in V$ and $w_0 \in W$, such that $a = \phi(v_0) + \varphi(w_0)$. Since k has at least three elements and $\bar{\phi}$ is universal by dimension reasons, there exists $\bar{v} \in \bar{V}$, such that $\bar{\phi}(\bar{v}) \neq 0 \neq \bar{a} - \bar{\phi}(\bar{v})$. By the universality of $\bar{\varphi}$ there exists then $\bar{w} \in \bar{W}$, such that $\bar{\varphi}(\bar{w}) = \bar{a} - \bar{\phi}(\bar{v})$. Let then $\bar{\tau}$ be an automorphism of q, such that $\bar{\tau}(\bar{v}_0, \bar{w}_0) = (\bar{v}, \bar{w})$, and $\tau \in \mathrm{O}(q)$ a preimage of $\bar{\tau}$ under the reduction map. The vector $(v, w) := \tau(v_0, w_0)$ has the desired properties. □

We begin now the proof of the main result of this section. Except for the use of Lemmas 2.2 and 2.3 above the arguments are almost word by word the same as in the field case (in fact even a bit shorter since these lemmas allow us to avoid some case by case considerations). For the sake of completeness (and to convince the reader of the correctness of the results) we give the details.

We start with the following lemma which corresponds to Elman and Lam [10, Prop. 2.2].

2.4. Lemma. *Let $q = \ll a_1, \ldots, a_n \gg$ be a n-Pfister form over R, $n \geq 2$. If $-b \in \mathrm{D}(q')^{\times}$, where q' is the pure subform of q, then there exist $b_2, \ldots, b_n \in R^{\times}$, such that $q \approx \ll b, b_2, \ldots, b_n \gg$.*

Proof. We prove this by induction on $n \geq 2$. The case $n = 2$ is Example 1.6 (i), so let $n \geq 3$. Set $\phi = \ll a_1, \ldots, a_{n-1} \gg$. Then by Witt cancellation we have $q' \simeq \phi' \perp < -a_n > \otimes \phi$. By Lemmas 2.2 and 2.3 we can assume that $-b = -x - a_n y$ and $y = t^2 - z$ with $-x, -z \in \mathrm{D}(\phi')^{\times}$ and $y \in \mathrm{D}(\phi)^{\times}$. By induction we have then

$$\ll x, b_2, \ldots, b_{n-1} \gg \approx \phi \approx \ll z, c_2, \ldots, c_{n-1} \gg$$

for some $b_i, c_i \in R^{\times}$, $2 \leq i \leq n-1$. As $< 1, -z >$ represents y we conclude then by Example 1.6 (ii) that $\ll z, a_n \gg \simeq \ll z, a_n y \gg$ and therefore

$$q = \phi \otimes \ll a_n \gg \approx \ll z, c_2, \ldots, c_{n-1}, a_n \gg \approx \ll z, c_2, \ldots, c_{n-1}, a_n y \gg$$

$$\approx \ll x, b_2, \ldots, b_{n-1}, a_n y \gg .$$

The pure subform of the 2-Pfister form $\ll x, a_n y \gg$ represents $-b$ and so by the case $n = 2$ we have $\ll x, a_n y \gg \simeq \ll b, b_n \gg$ for some $b_n \in R^{\times}$. We are done. □

The following two assertions correspond to [10, Cor. 2.5 and Thm. 2.6].

2.5. Lemma. *Let $p = \ll a_1, \ldots, a_m \gg$ and $q = \ll b_1, \ldots, b_n \gg$, $m, n \geq 1$ be two Pfister forms over the local ring R. Then:*

(i) *If $c \in \mathrm{D}(p)^{\times}$ then for all $d \in R^{\times}$ we have $p \otimes \ll d \gg \approx p \otimes \ll cd \gg$.*

(ii) *If $-c \in \mathrm{D}(p\otimes q')^\times$ then there are units $c_2,\ldots,c_n \in R$, such that*

$$p\otimes q \approx p\otimes \ll c, c_2,\ldots,c_n \gg \;.$$

Proof. We start with (i). If $m=1$ this follows form Example 1.6 (ii), so let $m\geq 2$. We can assume by Lemmas 2.2 and 2.3 that $c=t^2-z$ with $-z\in \mathrm{D}(p')^\times$, and so by the lemma above $p\otimes \ll d \gg$ is chain p-equivalent to $\ll z, c_2,\ldots,c_m, d\gg$ for some $c_i\in R^\times$, $2\leq i\leq m$. As $c\in \mathrm{D}(<1,-z>)^\times$ we have by Examples 1.6 (ii) the isometry $\ll z, d\gg \simeq \ll z, cd\gg$ from which the claim follows.

We prove now (ii) by induction on $n\geq 1$. The case $n=1$ is an immediate consequence of (i) as then $-c=-b_1\cdot x$ with $x\in \mathrm{D}(p)$. So let $n\geq 2$. Write $q=\phi\otimes \ll b_n\gg$, and so $p\otimes q' = p\otimes \phi' \perp < -b_n > \otimes p\otimes \phi$. Again by Lemmas 2.2 and 2.3 we can then write $-c=-x-b_n\cdot y$ with $-x\in \mathrm{D}(p\otimes\phi')^\times$ and $y\in \mathrm{D}(p\otimes \phi)^\times$. By induction we have then $p\otimes\phi\approx p\otimes \ll x, c_2,\ldots,c_{n-1}\gg$ for some $c_2,\ldots,c_{n-1}\in R^\times$. On the other hand since $y\in \mathrm{D}(p\otimes\phi)^\times$ we have by (i) that $p\otimes q = p\otimes\phi\otimes \ll b_n\gg \approx p\otimes \phi\otimes \ll b_n y\gg$, and so

$$p\otimes q \approx p\otimes \ll x, c_2,\ldots,c_{n-1}, b_n y\gg \;.$$

From this the claim follows since the pure subform of the 2-Pfister form $\ll x, b_n y\gg$ represents the unit $-c$, and so by Lemma 2.4 above we have $\ll x, b_n y\gg \simeq \ll c, c_n\gg$ for some $c_n\in R^\times$. □

We prove now the main result of this section.

2.6. Theorem. *Let $n\geq 2$, and $p=\ll a_1,\ldots,a_n\gg$ and $q=\ll b_1,\ldots,b_n\gg$ be two n-Pfister forms over the local ring R. Then*

$$p\simeq q \quad\Longleftrightarrow\quad p\approx q\,.$$

Proof. Obviously $p\approx q$ implies $p\simeq q$. For the other direction the case $n=2$ is by definition, so let $n\geq 3$. By Witt cancellation the pure subforms of p and q are isomorphic, and so $-b_1\in \mathrm{D}(p')^\times$.

Hence by Lemma 2.4 we have $p\approx \ll b_1, c_2,\ldots,c_n\gg$ for appropriate $c_2,\ldots,c_n\in R^\times$. Let $1\leq r\leq n$ be maximal, such that $p\approx \ll b_1,\ldots,b_r, d_{r+1},\ldots,d_n\gg$ for some units $d_{r+1},\ldots,d_n$ in R. We claim that $r=n$ which finishes the proof. Assume the contrary, *i.e.* $r<n$. We set then

$$\phi = \ll d_{r+1},\ldots,d_n\gg \qquad \text{and} \qquad \varphi = \ll b_{r+1},\ldots,b_n\gg \;.$$

Then we have

$$\ll b_1,\ldots,b_r\gg \otimes\phi \simeq \ll b_1,\ldots,b_r\gg\otimes\varphi\,,$$

and so by Witt cancellation we have $\ll b_1,\ldots,b_r\gg\otimes\phi' \simeq \ll b_1,\ldots,b_r\gg\otimes\varphi'$. Therefore $-b_{r+1}\in R^\times$ is represented by $\ll b_1,\ldots,b_r\gg\otimes\phi'$ and so it follows from Lemma 2.5 (ii) that

$$p\approx \ll b_1,\ldots,b_r, d_{r+1},\ldots,d_n\gg \approx \ll b_1,\ldots,b_r, b_{r+1}, e_{r+2},\ldots,e_n\gg$$

for some $e_{r+2},\ldots,e_n\in R^\times$, contradicting the maximality of r. We are done. □

3. Witt K-theory of a local ring

3.1. Convention. *In this section R denotes a local ring with residue field k. If R is not a field, i.e. $R \neq k$, we assume that k has at least 5 elements.*

3.2. *Witt K-theory of a local ring.* Witt K-theory of fields has been introduced by Morel [18]. Our definition for a local ring is the obvious and straightforward generalization.

Definition. The *Witt K-ring of R* is the quotient of the graded and free $\mathbb{Z}$-algebra generated by elements $[a]$ ($a \in R^\times$) in degree 1 and one element η in degree -1 by the two sided ideal, which is generated by the expressions

(WK1) $\eta \cdot [a] - [a] \cdot \eta$ with $a \in R^\times$;

(WK2) $[ab] - [a] - [b] + \eta \cdot [a] \cdot [b]$ with $a, b \in R^\times$;

(WK3) $[a] \cdot [1-a]$ with $a \in R^\times$, such that $1-a$ in $R^\times$; and

(WK4) $2 - \eta \cdot [-1]$.

We denote this graded $\mathbb{Z}$-algebra by $\mathrm{K}_*^W(R) = \bigoplus_{n\in\mathbb{Z}} \mathrm{K}_n^W(R)$. Note that by (WK2) the nth graded piece $\mathrm{K}_n^W(R)$ is generated by all products $[a_1] \cdot \ldots \cdot [a_n]$ for $n \geq 1$.

Following Morel [18] we set

$$< a >_{\mathrm{K}^W} :=\ 1 - \eta \cdot [a] \in \mathrm{K}_0^W(R)\,.$$

A straightforward computation using (WK1) and (WK2) shows that

$$< ab >_{\mathrm{K}^W} = < a >_{\mathrm{K}^W} \cdot < b >_{\mathrm{K}^W}$$

for all $a, b \in R^\times$. Note that using this notation (WK2) can be reformulated as

$$[ab] \;=\; [a] + < a >_{\mathrm{K}^W} \cdot [b]\,.$$

3.3. *Some elementary identities in $\mathrm{K}_*^W(R)$.* The following identities are proven for fields in Morel's article [18]. For the convenience of our reader we recall the rather short arguments which also work for local rings.

(1) By (WK4) we have $< -1 >_{\mathrm{K}^W} = -1$, and so using

$$< 1 >_{\mathrm{K}^W} = < -1 >_{\mathrm{K}^W} \cdot < -1 >_{\mathrm{K}^W}$$

we have $< 1 >_{\mathrm{K}^W} = 1$. The later equation implies then $[1] = 0$ since by (WK2) we have $[1] = [1 \cdot 1] = [1] + < 1 >_{\mathrm{K}^W} \cdot [1]$.

It follows from this that $< a >_{\mathrm{K}^W}$ is invertible with inverse

$$< a >_{\mathrm{K}^W}^{-1} = < a^{-1} >_{\mathrm{K}^W}$$

for all $a \in R^\times$.

(2) By (WK2) we have

$$\eta \cdot [a] \cdot [b] \;=\; \eta \cdot [b] \cdot [a]$$

for all $a, b \in R^\times$. This implies in particular that

$$< a >_{\mathrm{K}^W} \cdot [b] \;=\; [b] \cdot < a >_{\mathrm{K}^W}$$

for all $a, b \in R^\times$.

(3) We have
   (a) $[b^{-1}] \;=\; - < b^{-1} >_{\mathrm{K}^W} \cdot [b]$, and
   (b) $[ab^{-1}] = [a] - < ab^{-1} >_{\mathrm{K}^W} \cdot [b]$

for all $a, b \in R^\times$. In fact, we have by (1) above $[1] = 0$, and so $0 = [b^{-1} \cdot b] = [b^{-1}] + < b^{-1} >_{\mathrm{K}^W} \cdot [b]$, hence (a).

Since $< ab^{-1} >_{\mathrm{K}^W} = < a >_{\mathrm{K}^W} \cdot < b^{-1} >_{\mathrm{K}^W}$ we get from this

$$[a] - < ab^{-1} >_{\mathrm{K}^W} \cdot [b] = [a] + < a >_{\mathrm{K}^W} \cdot [b^{-1}] = [ab^{-1}]$$

as claimed.

The proof of the following lemma uses our assumption that if R is not a field then the residue field has at least 5 elements.

3.4. Lemma. *We have* $[-a] \cdot [a] = 0$ *for all* $a \in R^\times$.

Proof. The argument below is an adaption of the one in Nesterenko and Suslin [22] for Milnor K-theory of local rings.

Assume first that a, $1 - a$ and $1 - a^{-1}$ are units in R, *i.e.* the residue $\bar{a} \in k$ is not 1. Then $-a = \frac{1-a}{1-a^{-1}}$, and so we have

$$[-a] = [1 - a] - < -a >_{\mathrm{K}^W} \cdot [1 - a^{-1}]$$

using 3.3 (3) above. Therefore since $[1 - a][a] = 0$ we have

$$\begin{aligned} [-a] \cdot [a] &= - < -a >_{\mathrm{K}^W} \cdot [1 - a^{-1}] \cdot [a] \\ &= < -a >_{\mathrm{K}^W} \cdot [1 - a^{-1}] \cdot [a^{-1}] \cdot < a >_{\mathrm{K}^W} = 0\,, \end{aligned}$$

(using 3.3 (2) and (3)), and analogous $[a] \cdot [-a] = 0$. This proves the result in particular if R is a field.

Assume now that R is not a field and $\bar{a} = 1$ in the residue field k. By our assumption k has at least 5 elements. Let $c \in R^\times$, such that $\bar{c} \neq 1$ in k. Then also $\bar{a} \cdot \bar{c} \neq 1$ and so as we have shown already $[-ac] \cdot [ac] = [-c] \cdot [c] = [c] \cdot [-c] = 0$. This implies using (WK2) together with 3.3 (2)

$$[-a] \cdot [a] \;=\; -[c] \cdot [a] - [a] \cdot [c] + \eta \cdot \big([-a] \cdot [a] + [a] \cdot [a]\big) \cdot [c]\,. \tag{2}$$

Since $|k| \geq 5$ there exists $e, f \in R^\times$, such that $\bar{e}, \bar{f} \neq 1$ and also $\bar{e}\bar{f} \neq 1$. Applying Equation (2) to $c = ef$ and using 3.3 (2) we get

$$\begin{aligned} [-a] \cdot [a] &= -[ef] \cdot [a] - [a] \cdot [ef] + \eta \cdot \big([-a] \cdot [a] + [a] \cdot [a]\big) \cdot [ef] \\ &= -[e] \cdot [a] - [a] \cdot [e] + \eta \cdot \big([-a] \cdot [a] + [a] \cdot [a]\big) \cdot [e] \\ &\qquad -[f] \cdot [a] - [a] \cdot [f] + \eta \cdot \big([-a] \cdot [a] + [a] \cdot [a]\big) \cdot [f] \\ &\qquad\quad +2\eta \cdot [e] \cdot [f] \cdot [a] \; - \eta^2 \cdot ([-a] \cdot [a] + [a] \cdot [a]) \cdot [e] \cdot [f] \\ &= [-a] \cdot [a] + [-a] \cdot [a] \; + \eta \cdot [e] \cdot [f] \cdot [a] \cdot \big(2 - \eta \cdot [-a] + \eta \cdot [a]\big)\,, \end{aligned}$$

where the last equation is by (2) for $c = e, f$ and 3.3 (2). Now by (WK2) and 3.3 (2) we have

$$\begin{aligned} 2 - \eta \cdot [-a] - \eta \cdot [a] &= 2 - \eta \cdot [-1] - \eta \cdot [a] + \eta^2 \cdot [-1] \cdot [a] - \eta \cdot [a] \\ &= 2 - \eta \cdot [-1] + \eta \cdot [a] \cdot (-2 + \eta \cdot [-1])\,, \end{aligned}$$

which is 0 by (WK4). Hence $[-a] \cdot [a] = 0$. We are done. □

3.5. Corollary. *We have*

(i) $[a] \cdot [a] = [a] \cdot [-1]$ *for all* $a \in R^\times$*; and*

(ii) $[ab^2] = [a]$, *and so in particular* $[b^2] = 0$ *for all* $a, b \in R^\times$.

Proof. This can be proven as in the field case, see [18]. For the sake of completeness we recall briefly the details.

For (i), we have $[-a] = [-1] + <-1>_{\mathrm{K}^W} \cdot [a] = [-1] - [a]$ by (WK2) and 3.3 (1), and so using the lemma above $0 = [a] \cdot [-a] = [a] \cdot [-1] - [a] \cdot [a]$.

To show (ii) it is by (WK2) enough to show that $[b^2] = 0$, but this is a straightforward consequence of (WK2), (WK4), 3.3 (2), and part (i) of the corollary. $\square$

Assertions (i), (ii), and (iv) of the following theorem are less straightforward to prove. Note that (ii) part (b) and (iv) are new even for R a field.

3.6. Theorem.

(i) $\mathrm{K}^W_*(R)$ *is commutative.*

(ii) (a) $<a>_{\mathrm{K}^W} + <b>_{\mathrm{K}^W} = <a+b>_{\mathrm{K}^W} + <ab(a+b)>_{\mathrm{K}^W}$, *and*

(b) $[a] + [b] = [a+b] + [ab(a+b)]$

for all $a, b \in R^\times$ *such that also* $a + b \in R^\times$.

(iii) $[r] \cdot [st] + [s] \cdot [t] = [rs] \cdot [t] + [r] \cdot [s]$ *for all* $r, s, t \in R^\times$.

(iv) $[a+b] \cdot [ab(a+b)] = [a] \cdot [b]$ *for all* $a, b \in R^\times$ *such that* $a + b \in R^\times$.

Proof. For (i) it is enough to show that $[a] \cdot [b] = [b] \cdot [a]$ for all $a, b \in R^\times$. By Lemma 3.4 we have $[ab] \cdot [-ab] = 0 = [a] \cdot [-a]$ and so using (WK2) we get

$$\begin{aligned} 0 = [ab] \cdot [-ab] &= \big([a] + <a>_{\mathrm{K}^W} \cdot [b]\big) \cdot \big([-a] + <-a>_{\mathrm{K}^W} \cdot [b]\big) \\ &= <a>_{\mathrm{K}^W} \cdot \big([b] \cdot [-a] + <-1>_{\mathrm{K}^W} \cdot [a] \cdot [b]\big) + <-a^2>_{\mathrm{K}^W} \cdot [b] \cdot [b]\,. \end{aligned}$$

Using $[-a] = [a] + <a>_{\mathrm{K}^W} \cdot [-1]$, $<-1>_{\mathrm{K}^W} = -1$ and $<1>_{\mathrm{K}^W} = 1$ by 3.3 (1), and that $<rs^2>_{\mathrm{K}^W} = <r>_{\mathrm{K}^W}$ by Corollary 3.5 (ii) this equation is equivalent to

$$0 = <a>_{\mathrm{K}^W} \cdot \big([b] \cdot [a] - [a] \cdot [b]\big) + [b] \cdot [-1] - [b] \cdot [b]\,.$$

Now $[b] \cdot [b] = [b] \cdot [-1]$ by Corollary 3.5 (i) and so we get

$$0 = <a>_{\mathrm{K}^W} \cdot \big([b] \cdot [a] - [a] \cdot [b]\big)\,.$$

As $<a>_{\mathrm{K}^W}$ is a unit in $\mathrm{K}^W_*(R)$ by 3.3 (1) this proves our claim.

Part (a) of (ii) can be proven as in Morel [18, Cor. 3.8] for a field: By (WK3) we have $<r>_{\mathrm{K}^W} + <1-r>_{\mathrm{K}^W} = 1 + <r(1-r)>_{\mathrm{K}^W}$ for $r \in R^*$ with $1 - r$ also a unit. Setting $r = \frac{a}{a+b}$ gives then the result using that $<rs^2>_{\mathrm{K}^W} = <r>_{\mathrm{K}^W}$ for all $r, s \in R^\times$ by Corollary 3.5 (ii).

For part (b) we set $c = a + b$. Then by (WK2) and (WK3), 3.3 (3), and Corollary 3.5 we get

$$\begin{aligned} [abc \cdot c^{-1}] = [ab] &= [abc^{-2}] = [ac^{-1}] + [bc^{-1}] \\ &= [a] - <ac^{-1}>_{\mathrm{K}^W} \cdot [c] + [b] - <bc^{-1}>_{\mathrm{K}^W} \cdot [c]\,. \end{aligned}$$

On the other hand by 3.3 (3) again we have also

$$[abc \cdot c^{-1}] = [abc] - <ab>_{\mathrm{K}^W} \cdot [c]\,.$$

Putting these two equations together we get using $<rs^2>_{\mathrm{K}^W} = <r>_{\mathrm{K}^W}$ that

$$
\begin{aligned}
&[abc]+[c]-[a]-[b] \\
&\qquad = < ab >_{\mathrm{K}^W} \cdot [c] - \big(< a >_{\mathrm{K}^W} + < b >_{\mathrm{K}^W} \big) \cdot < c >_{\mathrm{K}^W} \cdot [c] + [c] \\
&\qquad = < ab >_{\mathrm{K}^W} \cdot [c] + \big(< a >_{\mathrm{K}^W} + < b >_{\mathrm{K}^W} \big) \cdot [c] + [c]\,,
\end{aligned}
\tag{3}
$$

where the second equality follows since $< r > \cdot [r] = -[r]$ for all $r \in R^\times$ as a direct computation using Corollary 3.5 (i) shows.

By part (a) we have $< a >_{\mathrm{K}^W} + < b >_{\mathrm{K}^W} = < abc >_{\mathrm{K}^W} + < c >_{\mathrm{K}^W}$ and hence by (3)

$$
\begin{aligned}
[abc]+[c]-[a]-[b] &= < ab >_{\mathrm{K}^W} \cdot [c] + \big(< abc >_{\mathrm{K}^W} + < c >_{\mathrm{K}^W} \big) \cdot [c] + [c] \\
&= < ab >_{\mathrm{K}^W} \cdot [c] + \big(< ab >_{\mathrm{K}^W} + 1 \big) \cdot < c >_{\mathrm{K}^W} \cdot [c] + [c] \\
&= < ab >_{\mathrm{K}^W} \cdot [c] - < ab >_{\mathrm{K}^W} \cdot [c] - [c] + [c] \\
&= 0\,,
\end{aligned}
$$

as claimed.

We show (iii). This is a straightforward consequence of (WK2) and the fact that $\mathrm{K}^W_*(R)$ is commutative:

$$
\begin{aligned}
[rs]\cdot[t] + [r]\cdot[s] &= \big([s] + < s >_{\mathrm{K}^W} \cdot [r]\big)\cdot[t] + [r]\cdot[s] \\
&= [s]\cdot[t] + [r]\big([s] + < s >_{\mathrm{K}^W} \cdot [t]\big) \\
&= [s]\cdot[t] + [r]\cdot[st]\,.
\end{aligned}
$$

Finally we show (iv). Setting in (iii) $r = a+b$, $s = a(a+b)$, and $t = b$, and using that $[xy^2] = [x]$ for all $x, y \in R^\times$ by Corollary 3.5 (ii) we get

$$
[a+b]\cdot[ab(a+b)] + [a(a+b)]\cdot[b] \;=\; [a]\cdot[b] + [a+b]\cdot[a(a+b)]\,. \tag{4}
$$

By (WK2) and (WK3) we have

$$
[x]\cdot[(1-x)y] \;=\; [x]\cdot[1-x] + [x]\cdot[y] + \eta\cdot[x]\cdot[1-x]\cdot[y] \;=\; [x]\cdot[y]
$$

for all $x, y \in R^\times$, such that $1-x$ also a unit in R. This equation together with Corollary 3.5 (ii) and the fact that $\mathrm{K}^W_*(R)$ is commutative shows that (note that $1 - a(a+b)^{-1} = b(a+b)^{-1}$ is a unit in R)

$$
\begin{aligned}
[a(a+b)]\cdot[b] &= [a(a+b)^{-1}]\cdot[b] \\
&= [a(a+b)^{-1}]\cdot[(1-a(a+b)^{-1})b] \\
&= [a(a+b)]\cdot[b^2(a+b)^{-1}] \;=\; [a+b]\cdot[a(a+b)]\,.
\end{aligned}
$$

Inserting this identity into (4) proves the claimed identity. We are done. $\square$

3.7. *Witt K-theory and the powers of the fundamental ideal.* The *Witt algebra* of R is the $\mathbb{Z}$-graded $\mathrm{W}(R)$-algebra

$$
\underline{W}_*(R) \;:=\; \bigoplus_{n\in\mathbb{Z}} \underline{W}_n(R)\,,
$$

where $\underline{W}_n(R) = \mathrm{I}^n(R)$ (recall that by convention $\mathrm{I}^n(R) = \mathrm{W}(R)$ for $n \le 0$), with the obvious addition and multiplication, *i.e.* if $x \in \underline{W}_m(R) = \mathrm{I}^m(R)$ and $y \in \underline{W}_n(R) = \mathrm{I}^n(R)$ then $x \cdot y$ is the class of $x \otimes y$ in $\mathrm{I}^{m+n}(R) = \underline{W}_{m+n}(R)$. Following Morel [18] we set $\eta_W := < 1 > \in \underline{W}_{-1}(R) = \mathrm{W}(R)$. Then multiplication by η_W corresponds to the natural embedding of $\underline{W}_{n+1}(R)$ into $\underline{W}_n(R)$ for all $n \in \mathbb{Z}$.

We have $\ll u, 1-u \gg = 0$ for units u of R, such that also $1-u \in R^\times$,

$$< 1 > \otimes \ll -1 \gg = < 1, 1 > = 2 \in W(R)\,,$$

and $\ll ab \gg = \ll a \gg + \ll b \gg - \ll a, b \gg$ for all $a, b \in R^\times$. Therefore there is a well defined homomorphism of $\mathbb{Z}$-graded $\mathbb{Z}$-algebras

$$\Theta_*^R : \mathrm{K}_*^W(R) \longrightarrow \underline{W}_*(R)\,, \quad \begin{array}{l} [u] \longmapsto \ll u \gg \in \underline{W}_1(R) = \mathrm{I}(R) \\ \eta \longmapsto \eta_W \in \underline{W}_{-1}(R) = \mathrm{W}(R)\,. \end{array}$$

Since $\mathrm{I}^n(R)$ for $n \geq 1$ is additively generated by n-Pfister forms and the $\mathbb{Z}$-module $\mathrm{W}(R)$ is generated by $< a > = \Theta_0^R(< a >_{\mathrm{K}^W})$, $a \in R^\times$, we see that Θ_n^R is surjective for all $n \geq 0$ and all local rings R. Our aim is to prove that Θ_*^R is an isomorphism for all "nice" regular local rings. More precisely, we show the following theorem, which is due to Morel [18] if R is a field.

3.8. Theorem. *Let R be a field or a local ring which contains an infinite field. Then*

$$\Theta_*^R : \mathrm{K}_*^W(R) \longrightarrow \underline{W}_*(R)$$

is an isomorphism.

4. Proof of Theorem 3.8

4.1. *A presentation of the powers of the fundamental ideal.* Our proof of Theorem 3.8 uses the following presentation of the powers of the fundamental ideal.

Theorem. *Let R be a field or a local ring whose residue field contains at least 5 elements. Let $\mathrm{PfM}_n(R)$ be the free abelian group generated by the isometry classes of n-Pfister forms over R. Following [13] we denote the isometry class of $\ll a_1, \ldots, a_n \gg$ by $[a_1, \ldots, a_n]$.*

(i) *The kernel of the homomorphism*

$$\mathbb{Z}[R^\times] \longrightarrow \mathrm{W}(R)\,,\ [r] \longmapsto < r >$$

is additively generated by $[1] - [-1]$, all $[ab^2] - [a]$ with $a, b \in R^\times$, and all expressions $[a]+[b]-([a+b]+[ab(a+b)])$ with $a, b \in R^\times$, such that $a+b \in R^\times$.

(ii) *The kernel of the homomorphism*

$$\mathbb{Z}[R^\times] \longrightarrow \mathrm{I}(R)\,,\ [r] \longmapsto \ll r \gg$$

is generated by $[1]$, all $[ab^2] - [a]$ with $a, b \in R^\times$, and all expressions of the form $[a] + [b] - ([a+b] + [ab(a+b)])$ with $a, b \in R^\times$, such that $a + b \in R^\times$.

(iii) *Assume now that R is a field or contains an infinite field of characteristic not 2. Then for $n \geq 2$ the kernel of the natural epimorphism*

$$\mathrm{PfM}_n(R) \longrightarrow \mathrm{I}^n(R)\,,\ [a_1, \ldots, a_n] \longmapsto \ll a_1, \ldots, a_n \gg$$

is generated by $[1, \ldots, 1]$; all sums

$$[a, c_2, \ldots, c_n] + [b, c_2, \ldots, c_n] - \big([a+b, c_2, \ldots, c_n] + [ab(a+b), c_2, \ldots, c_n]\big)$$

with $a, b \in R^\times$, such that $a + b \in R^\times$, and $c_2, \ldots, c_n \in R^\times$; and all sums

$$[a, b, d_3, \ldots, d_n] + [ab, c, d_3, \ldots, d_n] - \big([b, c, d_3, \ldots, d_n] + [a, bc, d_3, \ldots, d_n]\big)$$

with $a, b, c, d_3, \ldots, d_n \in R^\times$.

Proof. For R a local ring which contains an infinite field this has been proven by the first author [13]. For R a field this is due to Witt [29] if $n = 0, 1$ and to Arason and Elman [3, Thm. 3.1] if $n \geq 2$. Note that the latter work as well as [13] use the Milnor conjectures which are now theorems by the work of Voevodsky [28] and Orlov, Vishik, and Voevodsky [23]. □

Remarks.

(i) Morel [18, Lem. 3.10] claims that if R is a field, then in the above presentation of $\mathrm{I}(R)$, the so-called Witt relation $[a]+[b] = [a+b]+[ab(a+b)]$ can be replaced by the relation $[a] + [1-a] = [a(1-a)]$, $a \neq 0, 1$. As Detlev Hoffmann has pointed out to us the Laurent field in one variable over $\mathbb{Z}/3\mathbb{Z}$ is a counter example to this assertion.

(ii) Morel's [18] proof of Theorem 3.8 for a field R uses a slight alteration of Arason and Elman's [3, Thm. 3.1], the assertion [18, Thm. 4.1]. We do not know whether this follows from Arason and Elman [3, Thm. 3.1]. In any case, one can fixed this flaw using our corollary to Theorem 4.4 below.

We start now the proof of Theorem 3.8.

4.2. *The case* $n \leq 1$. Parts (i) and (ii) of the theorem above together with 3.3 (1), Lemma 3.4, Corollary 3.5, and Theorem 3.6 (i) and (ii) show that the morphisms

$$\underline{W}_0(R) \,=\, \mathrm{W}(R) \,\longrightarrow\, \mathrm{K}_0^W(R)\,, \quad < a > \longmapsto < a >_{\mathrm{K}^W}$$

and

$$\underline{W}_1(R) \,=\, \mathrm{I}(R) \,\longrightarrow\, \mathrm{K}_1^W(R)\,, \quad \ll a \gg \longmapsto [a]$$

are well defined. These are obviously inverse to Θ_0^R and Θ_1^R, respectively.

It follows from this that Θ_n^R is also an isomorphism for $n < 0$. In fact, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{K}_{n+1}^W(R) & \xrightarrow{\cdot\eta} & \mathrm{K}_n^W(R) \\
{\scriptstyle \Theta_{n+1}^R}\downarrow & & \downarrow{\scriptstyle \Theta_n^R} \\
\underline{W}_{n+1}(R) & \xrightarrow[\cdot\eta_W]{\simeq} & \underline{W}_n(R)
\end{array}$$

for all $n < 0$. By induction we can assume that Θ_{n+1}^R is an isomorphism. Since $\mathrm{K}_{n+1}^W(R) \xrightarrow{\cdot\eta} \mathrm{K}_n^W(R)$ is an epimorphism for all $n < 0$ this implies that also Θ_n^R is an isomorphism.

4.3. *The case* $n = 2$. The following argument is an adaption of a trick of Suslin [27, Proof of Lem. 6.3 (iii)] to Witt K-theory.

Consider the product of sets

$$\mathrm{K}_2^W(R) \times\, R^\times/(R^\times)^2\,.$$

We define an addition on this set by the rule

$$(x, \bar{r}) + (y, \bar{s}) \,:=\, (x + y + [r]\cdot[s], \bar{r}\cdot\bar{s})\,,$$

where $\bar{r}$ denotes the class of $r \in R^\times$ in $R^\times/(R^\times)^2$. As $[ab^2] = [a]$ in $\mathrm{K}_1^W(R)$ by Corollary 3.5 (ii) this is well defined. It is straightforward to check that with this addition the product of sets $\mathrm{K}_2^W(R) \times\, R^\times/(R^\times)^2$ is an abelian group with $(0, \bar{1})$ as zero and inverse $-(x, \bar{r}) = (-x - [r]\cdot[r^{-1}], \bar{r}^{-1})$.

Using part (ii) of the theorem in 4.1 together with 3.3 (1), Corollary 3.5 (ii), and Theorem 3.6 (iv) we see that

$$\mathrm{I}(R) \longrightarrow \mathrm{K}_2^W(R) \times R^\times/(R^\times)^2\,, \quad \ll r \gg \longmapsto (0,\bar{r})$$

is a well defined homomorphism. The image of the restriction of this map to $\mathrm{I}^2(R) \subseteq \mathrm{I}(R)$ is contained in the subgroup

$$\{\,(x,\bar{1})\,|\,x \in \mathrm{K}_2^W(R)\,\}$$

of $\mathrm{K}_2^W(R) \times R^\times/(R^\times)^2$ which is naturally isomorphic to $\mathrm{K}_2^W(R)$ via $(x,\bar{1}) \mapsto x$. The induced homomorphism $\mathrm{I}^2(R) \longrightarrow \mathrm{K}_2^W(R)$ maps $\ll r,s \gg$ to $[r]\cdot[s]$ and is therefore inverse to $\Theta_2^R : \mathrm{K}_2^W(R) \longrightarrow \underline{W}_2(R) = \mathrm{I}^2(R)$.

Altogether we have shown:

4.4. Theorem. *Let R be a field or a local ring whose residue field has at least 5 elements. Then*

$$\Theta_n^R \,:\, \mathrm{K}_n^W(R) \longrightarrow \underline{W}_n(R) \,=\, \mathrm{I}^n(R)$$

is surjective for all $n \in \mathbb{Z}$ and an isomorphism for all $n \leq 2$.

Corollary. *Let R be a field or a local ring whose residue field has at least 5 elements, and $n \geq 1$. Then*

$$\ll a_1,\ldots,a_n \gg \simeq \ll b_1,\ldots,b_n \gg$$

if and only if

$$[a_1]\cdot\ldots\cdot[a_n] \,=\, [b_1]\cdot\ldots\cdot[b_n]$$

in $\mathrm{K}_n^W(R)$ for all $a_1,\ldots,a_n,b_1,\ldots,b_n \in R^\times$. In particular, $[a_1]\cdot\ldots[a_n] = 0$ in $\mathrm{K}_n^W(R)$ if and only if $\ll a_1,\ldots,a_n \gg = 0$ in $\mathrm{W}(R)$.

Proof. If $n = 1$ this is obvious by the theorem above, so let $n \geq 2$. By Theorem 2.6 we know that isometric n-Pfister forms are chain p-equivalent and so we are reduced to the case $n = 2$, where the assertion follows again from the theorem above. □

With this corollary we can finish the proof of Theorem 3.8.

4.5. *End of the proof of Theorem 3.8.* Let now $n \geq 3$ and R be a field or a local ring which contains an infinite field. It follows from the corollary to Theorem 4.4 above that the homomorphism

$$\mathrm{PfM}_n(R) \longrightarrow \mathrm{K}_n^W(R)\,, \;\; [a_1,\ldots,a_n] \longmapsto [a_1]\cdot\ldots\cdot[a_n]$$

is well defined. Part (iii) of the theorem in 4.1 together with 3.3 (1) and Theorem 3.6 (ii) and (iii) shows that this homomorphism factors through $\mathrm{I}^n(R) = \underline{W}_n(R)$. It is obviously inverse to Θ_n^R. We are done.

5. Milnor-Witt K-theory of a local ring

5.1. Definition. Let R be a local ring. The *Milnor-Witt K-ring of R* is quotient of the graded and free $\mathbb{Z}$-algebra generated by elements $\{a\}$, $a \in R^\times$, in degree 1 and one element $\hat{\eta}$ in degree -1 by the two sided ideal which is generated by the expressions

(MW1) $\hat{\eta}\cdot\{a\} - \{a\}\cdot\hat{\eta}$, $a \in R^\times$;

(MW2) $\{ab\} - \{a\} - \{b\} + \hat{\eta}\cdot\{a\}\cdot\{b\}$, $a,b \in R^\times$;

(MW3) $\{a\}\cdot\{1-a\}$, if a and $1-a$ in $R^\times$; and

(MW4) $\hat{\eta}\cdot\big(2+\hat{\eta}\cdot\{-1\}\big)$.

We denote this graded $\mathbb{Z}$-algebra by $\mathrm{K}_*^{MW}(R)=\bigoplus\limits_{n\in\mathbb{Z}}\mathrm{K}_n^{MW}(R)$. Note that by (MW2) for $n\geq 1$ the nth graded piece $\mathrm{K}_n^{MW}(R)$ is generated as abelian group by all products $\{a_1,\ldots,a_n\}=\{a_1\}\cdot\ldots\cdot\{a_n\}$.

We prove now our main result about Milnor-Witt K-theory of local rings following Morel's arguments [18, Sect. 5] in the field case.

5.2. *Milnor K-theory, Witt K-theory, and Milnor-Witt K-theory.* Let R be a local ring. Milnor K-theory of a field has been introduced by Milnor [17]. The definition for rings we use here is the naive generalization of the one by Milnor and seems to have appeared for the first time in Nesterenko and Suslin [22].

Denote by $\mathrm{T}_{\mathbb{Z}}(R^\times)$ the tensor algebra of the abelian group $R^\times$ over $\mathbb{Z}$. The *Milnor K-theory* of R is the quotient of this algebra by the ideal generated by all tensors $a\otimes(1-a)$ with a and $1-a$ units in R. This is a graded $\mathbb{Z}$-algebra which we denote $\mathrm{K}_*^M(R)=\bigoplus\limits_{n\geq 0}\mathrm{K}_n^M(R)$. The class of a tensor $a_1\otimes\ldots\otimes a_n$ will be denoted as in the original source Milnor [17] by $\ell(a_1)\cdot\ldots\cdot\ell(a_n)$ as we have the now usual $\{a_1,\ldots,a_n\}$ reserved for symbols in Milnor-Witt K-theory.

There is a natural surjective homomorphism of graded $\mathbb{Z}$-algebras

$$\varpi_*^R\,:\,\mathrm{K}_*^{MW}(R)\longrightarrow\mathrm{K}_*^M(R)\,,$$

which maps $\hat{\eta}$ to 0 and $\{a\}$ to $\ell(a)$, $a\in R^\times$. The ideal $\hat{\eta}\cdot\mathrm{K}_{*+1}^{MW}(R)$ is in the kernel, and the induced homomorphism $\mathrm{K}_n^{MW}(R)/\hat{\eta}\cdot\mathrm{K}_{n+1}^{MW}(R)\longrightarrow\mathrm{K}_*^M(R)$ is an isomorphism. The inverse maps the symbol $\ell(a_1)\cdot\ldots\cdot\ell(a_n)$ to $\{a_1\}\cdot\ldots\cdot\{a_n\}$ modulo $\hat{\eta}\cdot\mathrm{K}_{n+1}^{MW}(R)$.

Set now $h:=2+\hat{\eta}\cdot\{-1\}\in\mathrm{K}_0^{MW}(R)$. Then by (MW4) we have $\hat{\eta}\cdot h=0$ and so there is an exact sequence

$$\mathrm{K}_{n+1}^{MW}(R)/h\cdot\mathrm{K}_{n+1}^{MW}(R)\xrightarrow{\cdot\hat{\eta}}\mathrm{K}_n^{MW}(R)\xrightarrow{\cdot\varpi_n^R}\mathrm{K}_n^M(R)\longrightarrow 0$$

for all $n\in\mathbb{Z}$, where $\mathrm{K}_n^M(R)=0$ for $n<0$ is understood. The group on the left hand side of this sequence is isomorphic to $\mathrm{K}_n^W(R)$. In fact, $\hat{\eta}\mapsto\eta$ and $\{u\}\mapsto-[u]$, $u\in R^\times$, defines a morphism of $\mathbb{Z}$-graded $\mathbb{Z}$-algebras $\mathrm{K}_*^{MW}(R)\longrightarrow\mathrm{K}_*^W(R)$, whose kernel contains the ideal generated by $h\in\mathrm{K}_0^{MW}(R)$, *i.e.* we have a homomorphism of $\mathbb{Z}$-graded $\mathbb{Z}$-algebras $\mathrm{K}_*^{MW}(R)/h\cdot\mathrm{K}_*^{MW}(R)\longrightarrow\mathrm{K}_*^W(R)$. This is an isomorphism. The inverse maps $[u]$ to $-\{u\}$ and η to $\hat{\eta}$ modulo $h\cdot\mathrm{K}_*^{MW}(R)$. Hence we have an exact sequence

$$\mathrm{K}_{n+1}^W(R)\xrightarrow{\epsilon_n^R}\mathrm{K}_n^{MW}(R)\xrightarrow{\varpi_n^R}\mathrm{K}_n^M(R)\longrightarrow 0 \tag{5}$$

for all $n\in\mathbb{Z}$, where ϵ_n^R maps $\eta^r\cdot[u_1]\cdot\ldots\cdot[u_{n+r}]$ to $(-1)^{n+r+1}\hat{\eta}^{r+1}\cdot\{u_1,\ldots,u_{n+r}\}$.

5.3. *Milnor-Witt K-theory and the powers of the fundamental ideal.* Let R be a local ring. As for Witt K-theory there is also a homogeneous homomorphism of $\mathbb{Z}$-graded $\mathbb{Z}$-algebras

$$\Upsilon_*^R\,:\,\mathrm{K}_*^{MW}(R)\longrightarrow\underline{W}_*(R)\,,\qquad\begin{aligned}\{u\}&\longmapsto-\ll u\gg\in\underline{W}_1(R)=\mathrm{I}(R)\\ \hat{\eta}&\longmapsto\eta_W\ \in\underline{W}_{-1}(R)=\mathrm{W}(R)\,.\end{aligned}$$

We leave it to the reader to check that this map is well defined. It is also surjective as $\Theta_*^R : \mathrm{K}_*^W(R) \longrightarrow \underline{W}_*(R)$ and Υ_*^R have the same image.

These maps fit into the following commutative diagram with exact rows

$$\begin{tikzcd}
& \mathrm{K}_{n+1}^W(R) \arrow[r, "\epsilon_n^R"] \arrow[d, "\Theta_{n+1}^R"'] & \mathrm{K}_n^{MW}(R) \arrow[r, "\varpi_n^R"] \arrow[d, "\Upsilon_n^R"'] & \mathrm{K}_n^M(R) \arrow[r] \arrow[d, "e_n^R"] & 0 \\
0 \arrow[r] & \mathrm{I}^{n+1}(R) \arrow[r, "\subseteq"'] & \mathrm{I}^n(R) \arrow[r] & \mathrm{I}^n(R)/\,\mathrm{I}^{n+1}(R) \arrow[r] & 0\ ,
\end{tikzcd} \tag{6}$$

where

$e_n^R : \mathrm{K}_n^M(R) \;\longrightarrow\; \mathrm{I}^n(R)/\,\mathrm{I}^{n+1}(R)\,,\;\; \ell(a_1)\cdot\ldots\cdot\ell(a_n) \;\longmapsto \ll a_1,\ldots,a_n \gg + \mathrm{I}^{n+1}(R)\,.$

By Theorems 3.8 and 4.4 we conclude by a diagram chase on (6) the following result.

5.4. Theorem. *Let R be a field or a local ring whose residue field has at least 5 elements. Then*

$$\begin{tikzcd}
\mathrm{K}_n^{MW}(R) \arrow[r, "\varpi_n^R"] \arrow[d, "\Upsilon_n^R"'] & \mathrm{K}_n^M(R) \arrow[d, "e_n^R"] \\
\mathrm{I}^n(R) \arrow[r] & \mathrm{I}^n(R)/\,\mathrm{I}^{n+1}(R)
\end{tikzcd}$$

is a pull-back diagram in the following cases:

(a) *$n \le 1$; or*

(b) *n is arbitrary and R is a field or a local ring which contains an infinite field.*

In particular, we have natural isomorphisms $\mathrm{K}_0^{MW}(R) \simeq \mathrm{GW}(R)$ (by the lemma in 1.7) and $\mathrm{K}_n^{MW}(R) \simeq \mathrm{W}(R)$ for all $n < 0$, and if R is a field or a regular local ring which contains an infinite field then the sequence

$$0 \longrightarrow \mathrm{K}_{n+1}^W(R) \xrightarrow{\epsilon_n^R} \mathrm{K}_n^{MW}(R) \xrightarrow{\varpi_n^R} \mathrm{K}_n^M(R) \longrightarrow 0$$

is exact for all $n \in \mathbb{Z}$.

6. Unramified Milnor Witt K-groups

6.1. *Residue maps.* Let F be a field with discrete valuation ν and residue field $F(\nu)$. Denote by π a uniformizer for ν. As well known, there is a so called (second) residue homomorphism

$$\partial_W^{\nu,\pi} \;:\; \mathrm{W}(F) \;\longrightarrow\; \mathrm{W}(F(\nu))\,,\; < u\pi^i > \longmapsto\; i\cdot < \bar{u} >\,,$$

where u is a ν-unit, $i \in \{0,1\}$, and $\bar{u}$ denotes the image of u in the residue field $F(\nu)$. This homomorphism depends on the choice of the uniformizer π, but its kernel is independent of this choice. It has been shown by Milnor [17, §5] that $\partial_W^{\nu,\pi}(\mathrm{I}^n(F) \subseteq \mathrm{I}^{n-1}(F)$ for all $n \in \mathbb{N}$. Let $\underline{\iota}_n(R) := \mathrm{I}^n(R)/\,\mathrm{I}^{n+1}(R)$. Then $\partial_W^{\nu,\pi}$ induces a residue homomorphism

$$\partial_{\underline{\iota}}^{\nu} \;:\; \underline{\iota}_n(F) \;\longrightarrow\; \underline{\iota}_{n-1}(F(\nu))$$

which does not depend on the choice of π.

Similarly there is a (second) residue homomorphism ∂^{ν}_{M}:

$$\mathrm{K}^M_n(F) \longrightarrow \mathrm{K}^M_{n-1}(F(\nu))\,,\ \ell(u\pi^i)\cdot\ell(u_2)\cdot\ldots\cdot\ell(u_n) \longmapsto i\cdot\ell(\bar u_2)\cdot\ldots\cdot\ell(\bar u_n)\,,$$

where $u, u_2, \ldots, u_n$ are ν-units and $i \in \{0,1\}$. This map does not depend on π.

Since $\mathrm{K}^{MW}_n(F)$ is the pull-back of $\mathrm{I}^n(F)$ and $\mathrm{K}^M_n(F)$ over $\underline{\iota}_n(F)$ by Theorem 5.4, these residue maps induce a *residue homomorphism* on Milnor-Witt K-theory

$$\partial^{\nu,\pi}_{MW}\ :\ \mathrm{K}^{MW}_n(F) \longrightarrow \mathrm{K}^{MW}_{n-1}(F(\nu))\,,$$

which is η_{MW}-linear and is uniquely determined by

$$\partial^{\nu,\pi}_{MW}(\{\pi, u_2, \ldots, u_n\}) \;=\; \{\bar u_2, \ldots, \bar u_n\}$$

and $\partial^{\nu,\pi}_{MW}(\{u_1, u_2, \ldots, u_n\}) = 0$, where $u_1, \ldots, u_n$ are ν-units. Hence it coincides with the residue map constructed by Morel [20, Thm. 3.15]. It depends on the choice of π, but the kernel does not.

6.2. *Unramified Milnor-Witt K-groups.* Let X be an integral locally noetherian scheme which is regular in codimension one. Then $\mathcal{O}_{X,x}$ is a discrete valuation ring for all $x \in X^{(1)}$, the set of points of codimension one in X. Choose an uniformizer $\pi_x \in \mathcal{O}_{X,x}$ for all such x. Then we have residue maps

$$\partial^{\pi_x}_{MW}\ :\ \mathrm{K}^{MW}_n(F(X)) \longrightarrow \mathrm{K}^{MW}_{n-1}(F(x))\,,$$

where $F(X)$ is the function field of X and $F(x)$ the residue field of $x \in X^{(1)}$. The nth *unramified Milnor-Witt K-group* of X is defined as

$$\mathrm{K}^{MW}_{n,unr}(X) \;:=\; \mathrm{Ker}\Big(\ \mathrm{K}^{MW}_n(F(X)) \xrightarrow{\ \sum_{x\in X^{(1)}}\partial^{\pi_x}_{MW}\ } \bigoplus_{x\in X^{(1)}} \mathrm{K}^{MW}_{n-1}(F(x))\Big)\,.$$

The unramified groups $\mathrm{W}_{unr}(X)$, $\mathrm{I}^n_{unr}(X)$, $\underline{\iota}_{n,unr}(X)$, and $\mathrm{K}^M_{n,unr}(X)$ are defined analogously. If $X = \mathrm{Spec}\,R$ is affine we use $\mathrm{K}^{MW}_{n,unr}(R)$ instead of $\mathrm{K}^{MW}_{n,unr}(X)$, $\mathrm{W}_{unr}(R)$ instead of $\mathrm{W}_{unr}(X)$ and so on.

Assume now that X is the spectrum of a regular local ring R which contains an infinite field. Denote by K the quotient field of R. It has been shown in [5] that $\mathrm{W}(R) \longrightarrow \mathrm{W}(K)$ induces an isomorphism $\mathrm{W}(R) \xrightarrow{\simeq} \mathrm{W}_{unr}(R)$. Kerz and Müller-Stach [16, Cor. 0.5] have shown $\mathrm{I}^n(K)\cap\mathrm{W}_{unr}(R) = \mathrm{I}^n(R)$ for all integers n and therefore we have also natural isomorphisms $\mathrm{I}^n(R) \xrightarrow{\simeq} \mathrm{I}^n_{unr}(R)$ and $\underline{\iota}_n(R) \xrightarrow{\simeq} \underline{\iota}_{n,unr}(R)$. On the other hand, the main result of Kerz [15] asserts that $\mathrm{K}^M_n(R) \longrightarrow \mathrm{K}^M_{n,unr}(R)$ is also an isomorphism for all $n \in \mathbb{N}$. These isomorphisms together with our Theorem 5.4 imply the following result.

6.3. Theorem. *Let R be a regular local ring which contains an infinite field. Then the natural homomorphism $\mathrm{K}^{MW}_n(R) \longrightarrow \mathrm{K}^{MW}_{n,unr}(R)$ is an isomorphism, i.e. the sequence*

$$0 \longrightarrow \mathrm{K}^{MW}_n(R) \longrightarrow \mathrm{K}^{MW}_n(K) \xrightarrow{\ \sum_{\mathrm{ht}\,P=1}\partial^{\pi_P}_{MW}\ } \bigoplus_{\mathrm{ht}\,P=1} \mathrm{K}^{MW}_{n-1}(R_P/PR_P)\,,$$

where K is the quotient field of R, is exact for all integers n.

As observed by Colliot-Thélène [9] this has the following consequence.

6.4. Corollary. *The nth unramified Milnor Witt K-group is a birational invariant of smooth and proper F-schemes for all integers n and infinite fields F.*

Proof. For the convenience of our reader we recall briefly Colliot-Thélène's argument. Let $f : X \dashrightarrow Y$ be a birational morphism between smooth and proper F-schemes. By symmetry it is enough to show that the induced homomorphism $f^* : \mathrm{K}_n^{MW}(F(Y)) \longrightarrow \mathrm{K}_n^{MW}(F(X))$ maps unramified elements to unramified elements. To see this we observe first that since Y is proper we can assume that f is defined on an open $U \subseteq X$ which contains $X^{(1)}$. Let now $x \in X^{(1)}$ and $y = f(x)$. Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{K}_n^{MW}(F(Y)) \arrow[r, "f^*"] & \mathrm{K}_n^{MW}(F(X)) \\
\mathrm{K}_n^{MW}(\mathcal{O}_{Y,y}) \arrow[u] \arrow[r, "f^*"'] & \mathrm{K}_n^{MW}(\mathcal{O}_{X,x}) \arrow[u]
\end{tikzcd}$$

from which the claim follows by the theorem above. □

6.5. A remark on Witt groups of schemes. The same method applied to Witt groups shows that the unramified Witt group of a smooth and proper scheme over a field of characteristic not 2 is also a birational invariant. This implication of the Gersten conjecture for Witt groups was already observed by Colliot-Thélène in [9] but at that time the Gersten conjecture was only known for low dimensional regular local rings. Only later Balmer [4] proved it for regular local rings of geometric type and Balmer, Panin, Walter and the first author [5] proved it for regular local rings containing a field (of characteristic not 2).

We finally mention the following application of the fact that the unramified Witt group is a birational invariant of smooth and proper schemes over a field of characteristic $\neq 2$, which follows from this by the Balmer-Walter [6] spectral sequence.

Theorem. *The Witt group is a birational invariant of smooth and proper schemes of dimension ≤ 3 over a field of characteristic not 2.*

By a theorem of Arason [2] we known that $W(F) \simeq \mathrm{W}(\mathbb{P}^n_F)$ for all positive integers n, and so

$$\mathrm{W}(F) \simeq \mathrm{W}(X)$$

for all rational smooth and proper F-schemes X of dimension ≤ 3.

References

[1] A. Asok, J. Fasel, *Splitting vector bundles outside the stable range and $\mathbb{A}^1$-homotopy sheaves of punctured affine spaces*, J. Amer. Math. Soc. (to appear).

[2] J. Arason, *Der Wittring projektiver Räume*, Math. Ann. **253** (1980), 205–212.

[3] J. Arason, R. Elman, *Powers of the fundamental ideal in the Witt ring*, J. Algebra **239** (2001), 150–160.

[4] P. Balmer, *Witt cohomology, Mayer-Vietoris, homotopy invariance and the Gersten conjecture*, K-Theory **23** (2001), 15–30.

[5] P. Balmer, S. Gille, I. Panin, Ch. Walter, *The Gersten conjecture for Witt groups in the equicharacteristic case*, Doc. Math. **7** (2002), 203–217.

[6] P. Balmer, Ch. Walter, *A Gersten-Witt spectral sequence for regular schemes*, Ann. Sci. École Norm. Sup. (4) **35** (2002), 127–152.

[7] J. Barge, F. Morel, *Cohomologie des groupes linéares, K-théorie de Milnor et grpupes de Witt*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), 287–290.

[8] J. Barge, F. Morel, *Groupe de Chow des cycles orientés et classe d'Euler des fibrés vectoriels*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), 191–196.

[9] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, in: K-Theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), 1–64, Proc. Sympos. Pure Math. **58**, Amer. Math. Soc., Providence, RI, 1995.

[10] R. Elman, T. Lam, *Pfister forms and K-theory of fields*, J. Algebra **23** (1972), 181–213.

[11] J. Fasel, *The Chow-Witt ring*, Doc. Math. **12** (2007), 275–312.

[12] J. Fasel, *Groupes de Chow-Witt*, Mém. Soc. Math. Fr. (N.S.) **113** (2008), 1–197.

[13] S. Gille, *Some consequences of the Panin-Pimenov Theorem*, Preprint 2015 (available at `www.math.ualberta.ca/~gille/publikationen1.html`).

[14] K. Hutchinson, L. Tao, *Homology stability for the special linear group of a field and Milnor-Witt K-theory*, Doc. Math. 2010, Extra volume: Andrei A. Suslin sixtieth birthday, 267–315.

[15] M. Kerz, *The Gersten conjecture for Milnor K-theory*, Invent. Math. **175** (2009), 1–33.

[16] M. Kerz, S. Müller-Stach, *The Milnor-Chow homomorphism revisited*, K-Theory **38** (2007), 49–58.

[17] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1969/70), 318–344.

[18] F. Morel, *Sur les puissances l'idéal fondamental de l'anneau de Witt*, Comment. Math. Helv. **79** (2004), 689–703.

[19] F. Morel, *On the motivic π_0 of the sphere spectrum*, Axiomatic, enriched and motivic homotopy theory, 219–260, NATO Sci. Ser. II Math. Phys. Chem. **131**, Kluwer Acad. Publ., Dordrecht 2004.

[20] F. Morel, *$\mathbb{A}^1$-algebraic topology over a field*, Lect. Notes Math. **2052**, Springer, Heidelberg, 2012.

[21] A. Neshitov, *Framed correspondences and the Milnor-Witt K-theory*, Preprint 2014 (avaible at arXiv:1410:7417).

[22] Y. Nesterenko, A. Suslin, *Homology of the full linear group over a local ring and Milnor's K-theory*, Math. USSR Izv. **34** (1990), 121–145.

[23] D. Orlov, A. Vishik, V. Voevodsky, *An exact sequence for $\mathrm{K}_*^M/2$ with applications to quadratic forms*, Ann. of Math. (2) **165** (2007), 1–13.

[24] I. Panin, K. Pimenov, *Rationally isotropic quadratic spaces are locally isotropic: II*, Doc. Math. 2010, Extra volume: Andrei A. Suslin sixtieth birthday, 515–523.

[25] W. Scharlau, *Quadratic and hermitian forms*, Springer-Verlag, Berlin, 1985.

[26] M. Schlichting, *Euler class groups, and the homology of elementary and special linear groups*, Preprint 2015 (avaible at arXiv:1502.05424).

[27] A. Suslin, *Torsion in K_2^M of fields*, K-Theory **1** (1987), 5–29.

[28] V. Voevodsky, *Motivic cohomology with $\mathbb{Z}/2$-coefficients*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 59–104.

[29] E. Witt, *Theorie der quadratischen Formen in beliebigen Körpern*, J. reine angew. Math. **176** (1937), 31–44.

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