ON AN EXTENSION OF HOFFMANN'S SEPARATION THEOREM FOR QUADRATIC FORMS

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Abstract. Let p and q be anisotropic non-degenerate quadratic forms of dimension ≥ 2 over an arbitrary field F, let s be the unique non-negative integer for which $2^s < \dim(p) \leq 2^{s+1}$, and let k be the dimension of the anisotropic part of q after extension to $F(p)$. A recent conjecture of the author then asserts that $\dim(q)$ must lie within k of an integer multiple of 2^{s+1}. This statement, which holds trivially if $k \geq 2^s - 1$, represents a natural generalization of the well-known separation theorem of Hoffmann, bridging a gap between it and certain classical results on the Witt kernels of function fields of quadrics. In the present article, we prove the conjecture in the case where $\mathrm{char}(F) \neq 2$ and $\dim(p) > 2k - 2^{s-1}$. This implies, in particular, that the conjecture holds if $\mathrm{char}(F) \neq 2$ and either $k \leq 2^{s-1} + 2^{s-2}$ or $\dim(p) \geq 2^s + 2^{s-1} - 4$.

1. Introduction

Let p and q be anisotropic non-degenerate quadratic forms of dimension ≥ 2 over a field F, and let s be the unique non-negative integer for which $2^s < \dim(p) \leq 2^{s+1}$. An important unifying problem in the algebraic theory of quadratic forms is to understand the extent to which q can become isotropic after scalar extension to the function field $F(p)$ of the projective quadric $\{p = 0\}$ (the extent of isotropy being measured by the Witt index $\mathfrak{i}_W(q_{F(p)})$). In the absence of any additional hypotheses, this problem is sufficiently complex that it already represents a challenge to determine the constraints coming from basic discrete invariants of the given forms, including the simplest invariants of all – the dimensions of p and q. To this end, we formulated in [20] the following general conjecture that aims to bridge a gap between two classic results in the literature:

Conjecture 1.1. *In the above situation, let* $k = \dim((q_{F(p)})_{\mathrm{an}}) = \dim(q) - 2\mathfrak{i}_W(q_{F(p)})$. *Then* $\dim(q) = a2^{s+1} + \epsilon$ *for some non-negative integer* a *and some* $-k \leq \epsilon \leq k$.

Loosely speaking, Conjecture 1.1 asserts that the more isotropic q becomes over $F(p)$, the closer its dimension must be to being divisible by 2^{s+1}. From a geometric viewpoint, this amounts to progressively severe dimension restrictions on the existence of rational maps between the quadric defined by p and the isotropic grassmannians associated to q. The two classic results alluded to above are visible at opposite extremities of the statement. First, in the case where $\dim(q) \leq 2^s$, the conjecture claims that $k = \dim(q)$, i.e., that q remains anisotropic over $F(p)$. This is precisely the statement of the so-called *separation theorem* initially discovered by Hoffmann in [6] (and later extended to characteristic 2 by Hoffmann and Laghribi in [7]). Second, in the case where $k = 0$ (i.e., where q becomes not merely isotropic, but *hyperbolic* over $F(p)$), the conjecture proposes that $\dim(q)$ is necessarily divisible by 2^{s+1}, a strong refinement of the basic inequality $\dim(q) \geq 2^{s+1}$ implied by the Cassels-Pfister subform theorem (which says that q must contain a subform similar to p in this case – see [2, Thm. 22.5]). We note that although the Cassels-Pfister

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subform theorem played a key role in Hoffmann's original proof of the separation theorem, the statements themselves had previously appeared somewhat isolated from one another.

In [20], the following enhancement of the preceding statement concerning the case of hyperbolicity was proved:[1] If q becomes hyperbolic over $F(p)$, then not only is $\dim(q)$ divisible by 2^{s+1}, but so too are all higher Witt indices of q, with the possible exception of the last (which is at least divisible by 2^s). This gives a satisfying explanation of another classic result of Fitzgerald ([3]) and perhaps raises new questions about the structure of the (poorly understood) kernel of the scalar extension map $W_q(F) \to W_q\big(F(p)\big)$ on Witt groups. Beyond the case of hyperbolicity, it was further shown[1] in [20] that Conjecture 1.1 holds if $k < 2^{s-1}$ or if $\dim(p) \in \{2^{s+1}-1, 2^{s+1}\}$. Since the statement of the conjecture holds trivially if $k \geq 2^s - 1$, this covers half of its non-trivial cases. In the present article, we make the following improvement to this result (at least when $\mathrm{char}(F) \neq 2$):

Theorem 1.2. *If* $\mathrm{char}(F) \neq 2$*, then Conjecture 1.1 holds in the case where* $\dim(p) > 2k - 2^{s-1}$*. In particular, if* $\mathrm{char}(F) \neq 2$*, then the conjecture holds when*

(1) $k \leq 2^{s-1} + 2^{s-2}$*, or*
(2) $2^s + 2^{s-1} - 4 \leq \dim(p) \leq 2^{s+1}$.

Note here that the condition $\dim(p) > 2k - 2^{s-1}$ is vacuous if (1) holds (because $\dim(p) > 2^s$) and effectively vacuous if (2) holds (since we may assume that $k \leq 2^s - 2$). The second statement of the theorem therefore follows immediately from the first. We also remark here that the assumption on the characteristic of F is imposed only out of a sense of caution. Indeed, the proof of Theorem 1.2 is mostly geometric, the main tool being the action of Steenrod operations on the mod-2 Chow groups of smooth projective varieties. For some time, the absence of such operations in the characteristic-2 setting would have rendered our assumption essential (this is the reason it appears in [20], for example). Thanks to the recent work of Primozic ([18]), however, this is no longer an issue. At the same time, the proof of Theorem 1.2 uses two results on quadratic forms that are not explicitly available in characteristic 2, namely, a result of Hoffmann on virtual Pfister neighbours ([6, Cor. 3]) and Rost's computation of the Chow groups of Rost motives ([19, Thm. 5]). While it is clear that both results remain valid in characteristic 2, some algebraic aspects of the existing proofs must be modified in the latter setting. We therefore choose to err on the side of caution and impose our restriction. Note, however, that the aforementioned results of [6, 19] are only needed for certain cases of Theorem 1.2. In particular, the characteristic assumption plays no role if $\dim(p) \geq 2^s + 2^{s-1}$ or $k \leq 2^{s-1}$ (the latter confirming that the main result of [20] is valid in characteristic 2) – see Remark 8.10. Along the way to proving Theorem 1.2, we also show that Conjecture 1.1 holds when $\dim(q) \leq 2^{s+2} + 2^{s+1} + k$, irrespective of the characteristic of F (see Proposition 6.1).

The proof of Theorem 1.2 follows an approach outlined in [20]. More specifically, let P and Q denote the projective quadrics defined by the vanishing of p and q, respectively. Over the field $F(p)$, the quadric Q contains a projective linear subspace of dimension $(\dim(Q) - k)/2$. Passing to Chow groups modulo 2, it turns out that Conjecture 1.1 can be reformulated as the statement that the fundamental class of this linear subspace is annihilated by a certain family of Steenrod operations. In [24], Vishik introduced a method for establishing the rationality of elements in the mod-2 Chow groups of smooth projective varieties over function fields of quadrics obtained via the action of Steenrod operations, after possibly modding out elements coming from integral torsion classes. A reinterpretation of this method that avoids the use of algebraic cobordism theory was given

[1] under the (now unnecessary) assumption that $\mathrm{char}(F) \neq 2$.

by Karpenko in [14]. Taking into account Springer's theorem on odd-degree extensions ([2, Cor. 18.5]), Conjecture 1.1 effectively proposes that [24, Thm 3.1] (which is optimal in general) admits a non-trivial refinement in the case where Vishik's variety Y is itself a quadric. To achieve the needed refinement in the situation of Theorem 1.2, a key step in our approach is to show that the relevant linear subspace class in $\mathrm{CH}(Q_{F(p)})/2$ can be lifted under the natural surjection $\mathrm{CH}(P\times Q)/2 \to \mathrm{CH}(Q_{F(p)})/2$ to a correspondence that enjoys a particular decomposition after extension to an algebraic closure of F (see Proposition 8.2). Note that in the general situation considered in [14, 24], one has no such control over the cycles under consideration. Here, the existence of the needed correspondence is obtained by examining the possible Chow motivic decompositions of P and Q. As part of this analysis, we give in §5 a splitting-pattern-theoretic criterion for the existence of binary direct summands in (internal shells of) the motives of quadrics, an observation which carries some independent interest (see Proposition 5.1). Another tool in the argument is Izhboldin's theorem on stable birational equivalence of quadrics ([8, Thm. 0.2], extended to characteristic 2 in [7]), which is itself a close descendent of the separation theorem. Once Proposition 8.2 is established, the last main step in the proof of Theorem 1.2 is to address a technical problem concerning the action of Steenrod operations on mod-2 reductions of torsion elements in the Chow group of P over a certain extension of F (see Propositions 7.1 and 8.7). The key ingredients here are the aformentioned computation of the Chow groups of Rost motives due to Rost, as well as our Proposition 5.1.

In [20, Ex. 1.5], it was shown that Conjecture 1.1 is optimal, in the sense that there are no further gaps in the possible values of $\dim(q)$ determined by the integers s and k alone. In the final section of the present article, however, we note that the statement should admit a refinement in the case where p is not a Pfister neighbour. More precisely, if p is not a Pfister neighbour, and $k < 2^{s-1} + 2^{s-2}$, then the conclusion of Conjecture 9.4 should hold with the integer 2^{s+1} replaced by 2^{s+2} (see Conjecture 9.4). While we do not prove this assertion, our methods allow us to show (with the aforementioned characteristic restrictions) that it holds if the condition that p is not a Pfister neighbour is replaced by the condition that the upper motive of the quadric P is not binary (Theorem 9.1). Note that these conditions are conjecturally equivalent by [23, Conj. 4.21].

Finally, we remark that if the integer k is redefined as $\dim(q) - 2\mathfrak{i}_0(q_{F(p)})$, with $\mathfrak{i}_0(q_{F(p)})$ being the maximal dimension of a totally isotropic subspace for $q_{F(p)}$, then we expect the statement of Conjecture 1.1 to remain valid irrespective of whether p and q are non-degenerate.[2] Evidence for this claim was given in [21], where the case where q is quasilinear was studied and settled in a large number of cases. Since the methods of the present article require the quadrics P and Q to be smooth, we will not deal with degenerate cases here.

Conventions, notation and terminology. If k is a field, then $\overline{k}$ will denote a fixed algebraic closure of k. By a *variety*, we shall mean a quasi-projective scheme of finite type over a field. Integral (resp. mod-2) Chow groups will be denoted with the letters CH (resp. Ch). All quadratic forms considered are assumed to be non-degenerate, meaning that their associated projective quadrics are smooth, or, equivalently, that their quasilinear parts are anisotropic of dimension ≤ 1 (see [2, §7.A.]). If φ is a quadratic form over a field k which is not isometric to the hyperbolic plane $\mathbb{H}$, then we will write $k(\varphi)$ for the function field of its associated (integral) projective quadric. To avoid case distinctions, we also set $k(\mathbb{H}) = k$. While we will recall some standard facts concerning Chow groups of (products of) quadrics in §§2,3 below, we assume basic familiarity with other aspects of

[2]In characteristic 2, anisotropic quadratic forms may be far from non-degenerate. If p is quasilinear and q is degenerate, then the integer $\mathfrak{i}_0(q_{F(p)})$ may be greater than the Witt index of $q_{F(p)}$ (as defined in [2]).

the algebraic theory of quadratic forms, including the generic splitting theory of Knebusch ([17]). Important results on function fields of quadrics such as the separation theorem and the determination of the values of the first higher Witt index ([12]; [18, Prop. 10.4] for its recent extension to characteristic 2) will be used frequently. For any other undefined notation or terminology concerning quadratic forms, we refer the reader to [2]. Note that while we will generally cite [2, Ch. XI] for properties of Steenrod operations on Chow groups modulo 2, [18] ensures the validity of all these results in characteristic 2.

2. Some notation, terminology and basic facts

Throughout this section we fix an arbitrary field k (not to be confused with the integer k from the statement of Conjecture 1.1). We consider only varieties X with the property that the scalar extension map $\mathrm{CH}(X_K) \to \mathrm{CH}(X_L)$ is an isomorphism for any field extension L/K with K algebraically closed. This holds if X is the product of finitely many smooth projective quadrics, the only case needed in the sequel (see, e.g., [2, §68]).

2.A. **Chow groups in the limit.** Given a smooth variety X over k (satisfying the aforementioned condition), we set

$$\mathrm{CH}(\overline{X}) = \mathrm{colim}_{K/k}\,\mathrm{CH}(X_K)$$

(where K/k runs over all field extensions of k). For any integer $i \geq 0$, the i-dimensional (resp. i-codimensional) part of $\mathrm{CH}(\overline{X})$ is denoted $\mathrm{CH}_i(\overline{X})$ $\big($resp. $\mathrm{CH}^i(\overline{X})\big)$. Note that these groups inherit the basic functorial properties of Chow groups – pull-backs along arbitrary morphisms and push-forwards along proper morphisms. In particular, $\mathrm{CH}(\overline{X})$ enjoys a ring structure compatible with pull-backs and subject to the usual projection formula for proper push-forwards. If K/k is a field extension, and $\beta \in \mathrm{CH}(X_K)$, then we write $\overline{\beta}$ for the image of β under the canonical homomorphism $\mathrm{CH}(X_K) \to \mathrm{CH}(\overline{X})$. An element of $\mathrm{CH}(\overline{X})$ equal to $\overline{\beta}$ for some $\beta \in \mathrm{CH}(X_K)$ will be said to be K*-rational.* Note that pull-backs and proper push-forwards preserve K-rationality for any field extension K/k. In the sequel, we will also work with mod-2 versions of the groups $\mathrm{CH}(\overline{X})$. In this case, the letters CH will be replaced by Ch, with the preceding terminology and notation pertaining to rationality issues being applied in the same way.

2.B. **Chow correspondences.** Let X and Y be smooth varieties over k. By a (*geometric*) *correspondence* from X to Y, we shall mean an element of the group $\mathrm{CH}(X\times Y)$ $\big($resp. $\mathrm{CH}(\overline{X \times Y})\big)$. For information on the standard composition law for correspondences between smooth projective varieties over fields, we refer to [2, §62]. Note that this law is compatible with scalar extension. In particular, if X, Y and Z are smooth projective varieties over k, then we have an induced group homomorphism

$$\mathrm{CH}(\overline{X \times Y}) \times \mathrm{CH}(\overline{Y \times Z}) \to \mathrm{CH}(\overline{X \times Z}), \quad (\alpha, \beta) \mapsto \beta \circ \alpha.$$

For a fixed $\beta \in \mathrm{CH}(\overline{Y \times Z})$, we will denote the group homomorphism

$$\mathrm{CH}(\overline{X \times Y}) \to \mathrm{CH}(\overline{X \times Z}), \quad \alpha \mapsto \beta \circ \alpha$$

by β_*. Note that if β is K-rational for some field extension K/k, then β_* maps K-rational elements to K-rational elements. Again, all these comments also apply for Chow groups modulo 2, and we will use the same notation and terminology in that context.

2.C. **Quadrics.** Let φ be a quadratic form of dimension ≥ 2 over k, let X be the (smooth) projective quadric defined by the vanishing of φ and let $n = [\dim(X)/2]$. For each $0 \leq i \leq n$, we write H^i for the class in $\mathrm{CH}^i(X)$ of a codimension-i plane section of X, and h^i for its image in $\mathrm{Ch}^i(X)$. If $\mathfrak{i}_W(\varphi) > i$, then we also write L_i for the class in $\mathrm{CH}_i(X)$ of a projective linear subspace of dimension i in X, and l_i for its mod-2 reduction. All these elements are independent of any of the indicated choices, with one exception in the case where φ is hyperbolic; in this case, there are two independent families of rationally equivalent projective linear subspaces of dimension n in X. In order to define the element L_n (or l_n), we must therefore choose one of these two classes; following [2, Ch. XIII], we say that we choose an *orientation* of X. In the sequel, we will suppress all choices of orientation from the discussion. When two or more quadrics (over possibly different fields) are under consideration and related through a given morphism or Chow correspondence, it is tacitly understood that compatible choices of orientations have been made. We recall the action of the cohomological Steenrod operations of [1, 18] on the mod-2 classes:

Lemma 2.1 (cf. [2, Cor. 78.5])**.** *Let $0 \leq i \leq n$. Then:*

(1) *$S^j(h^i) = \binom{i}{j} h^{i+j}$ for any $j \geq 0$;*
(2) *If $i < \mathfrak{i}_W(\varphi)$, then $S^j(l_i) = \binom{\dim(\varphi)-i-1}{j} l_j$ for any $0 \leq j \leq i$.*

For each $0 \leq i \leq n$, we make no notational distinction between the elements $H^i, L_i \in \mathrm{CH}(X_{\overline{k}})$ and their images under the canonical map $\mathrm{CH}(X_{\overline{k}}) \to \mathrm{CH}(\overline{X})$. The same applies to the mod-2 classes $h^i, l_i \in \mathrm{Ch}(X_{\overline{k}})$ and their images in $\mathrm{Ch}(\overline{X})$. The set $\{H^i, L_i \mid 0 \leq i \leq n\}$ (resp. $\{h^i, l_i \mid 0 \leq i \leq n\}$) constitutes a basis of $\mathrm{CH}(\overline{X})$ $\big($resp. $\mathrm{Ch}(\overline{X})\big)$ as an abelian group (resp. $\mathbb{F}_2$-vector space). For more details, including a description of the ring structures, see [2, §68].

2.D. **Degree maps for quadrics.** Let φ, X and n be as above. For each $0 \leq i \leq \dim(X)$, there is a unique group homomorphism $\overline{\deg}_i \colon \mathrm{CH}_i(\overline{X}) \to \mathbb{Z}$ that sends $H^{\dim(X)-i}$ to 2 and L_i to 1 (assuming that $i \leq n$). Composing with the canonical map $\mathrm{CH}_i(X) \to \mathrm{CH}_i(\overline{X})$, we obtain a homomorphism $\deg_i \colon \mathrm{CH}_i(X) \to \mathbb{Z}$. Note that $\deg_n$ does not depend on the choice of orientation of $X_{\overline{k}}$ in the case where $\dim(\varphi)$ is even.

Lemma 2.2. *In the above situation:*

(1) $\deg_0$ *is the usual degree homomorphism* $\mathrm{CH}_0(X) \to \mathbb{Z}$*;*
(2) *For any i and any $\alpha \in \mathrm{CH}_i(X)$, we have $\deg_i(\alpha) = \deg(H^i \cdot \alpha)$;*
(3) *For any i, the torsion subgroup of $\mathrm{CH}_i(X)$ lies in the kernel of $\deg_i$.*

Proof. (1) is clear, while (2) follows from (1) and the identities $H^i \cdot H^{\dim(X)-i} = H^{\dim(X)}$ and $H^i \cdot L_i = L_0$ in $\mathrm{CH}_0(\overline{X})$ (see [2, §68]). Part (3) holds since $\mathrm{CH}(\overline{X})$ is torsion free. $\square$

Note that $\deg_i\big(\mathrm{CH}_i(X)\big) = 2\mathbb{Z}$ for all $i > n$. The theorem of Springer on odd-degree extensions says that $\deg\big(\mathrm{CH}_0(X)\big) = 2\mathbb{Z}$ if and only if φ is anisotropic ([2, Cor. 71.3]).

2.E. **Isotropic quadrics.** Suppose now that the form φ given above is not split, and let Y be the projective quadric defined by the vanishing of its anisotropic kernel φ_{an}. The choice of a maximal totally isotropic subspace for φ yields an obvious k-rational incidence correspondence $in^X \in \mathrm{CH}(\overline{Y \times X})$ with the property that the induced homomorphism $(in^X)_* \colon \mathrm{CH}(\overline{Y}) \to \mathrm{CH}(\overline{X})$ sends H^i to H^{i+a} and L_i to L_{i+a} for all $0 \leq i \leq n-a$ (see [2, Lem. 72.3]). Letting pr^X denote the transpose of in^X (i.e., the element obtained from in^X by switching X and Y), we also have that $(pr^X)_* \colon \mathrm{CH}(\overline{X}) \to \mathrm{CH}(\overline{Y})$ sends H^i to H^{i-a} and L_i to L_{i-a} for all $a \leq i \leq n$ ([*loc. cit.*]). For any field extension K/k, the maps

$(in^X)_*$ and $(pr^X)_*$ send K-rational elements to K-rational elements. Obviously all of this also applies for the mod-2 Chow groups (with the elements $H^?$ and $L_?$ being replaced by $h^?$ and $l_?$, respectively). Using Springer's theorem and Lemma 2.2, we deduce:

Lemma 2.3 (cf. [2, Cor. 72.6]). *For any $0 \leq i \leq n$, the following are equivalent:*

1. $\mathfrak{i}_W(\varphi) \leq i$*;*
2. *The element $L_i \in \mathrm{CH}(\overline{X})$ is not F-rational;*
3. *The element $l_i \in \mathrm{Ch}(\overline{X})$ is not F-rational;*
4. $\deg_j\big(\mathrm{CH}_j(X)\big) = 2\mathbb{Z}$ *for all $j \geq i$.*

3. Cycles modulo 2 on products of quadrics

We continue to work over a fixed field k. Let φ be a quadratic form of dimension ≥ 2 over k, X its associated (smooth) projective quadric and $n = [\dim(X)/2]$. If Z is a smooth variety over k satisfying the condition stated at the beginning of §2, then the external product homomorphisms

$$\mathrm{Ch}(X_K) \otimes \mathrm{Ch}(Z_K) \to \mathrm{Ch}\big((X \times Z)_K\big)$$

(with K/k running over all field extensions of k) collectively determine a ring isomorphism

$$\mathrm{Ch}(\overline{X}) \otimes \mathrm{Ch}(\overline{Z}) \to \mathrm{Ch}(\overline{X \times Z})$$

(see [2, Prop. 64.3]). We again write $\alpha \times \beta$ for the image of a pure tensor $\alpha \otimes \beta$ under this isomorphism. In what follows, we will be interested only in the case where Z is a product of smooth projective quadrics. Before proceeding, we pause to note the following basic fact (recall here that $k(\varphi)$ denotes the function field of the (integral) quadric X in the case where $\varphi \not\cong \mathbb{H}$, and that $k(\varphi) = k$ in the case where $\varphi \cong \mathbb{H}$):

Lemma 3.1. *Let m be a non-negative integer and let $z \in \mathrm{Ch}^m(\overline{Z})$. Then the following are equivalent:*

1. $\mathrm{Ch}(\overline{X \times Z})$ *contains a k-rational element of the form*
$$(h^0 \times z) + \left(\sum_{i=1}^{n} a_i(h^i \times z_i)\right) + \left(\sum_{i=0}^{n} b_i(l_{n-i} \times z_i')\right)$$
for some $z_i \in \mathrm{Ch}^{m-i}(\overline{Z})$, $z_i' \in \mathrm{Ch}^{m-\dim(X)+n-i}(\overline{Z})$ and $a_i, b_i \in \mathbb{F}_2$;
2. *z is $k(\varphi)$ rational.*

Proof. If $\varphi \cong \mathbb{H}$, then the statement is clear. Otherwise, the statement follows from the surjectivity of the pull-back homomorphism

$$\mathrm{Ch}(X \times Z) \to \mathrm{Ch}(Z_{k(\varphi)})$$

given by restriction to the generic point of X (see [2, Cor. 57.11]), together with the descriptions of $\mathrm{Ch}(\overline{X})$ and $\mathrm{Ch}(\overline{X \times Z})$ given in §2.C and the preceding discussion. $\square$

3.A. **Products of quadrics.** Now let $\varphi_1, \ldots, \varphi_m$ be a collection of $m \geq 2$ (not necessarily distinct) quadratic forms of dimension ≥ 2 over k, and let $X_1, \ldots, X_m$ denote the (smooth) projective quadrics defined by the vanishing of the $\varphi_1, \ldots, \varphi_m$, respectively. By the remarks preceding Lemma 3.1, formation of external products gives a ring isomorphism

$$\bigotimes_{i=1}^{m} \mathrm{Ch}(\overline{X_i}) \to \mathrm{Ch}(\overline{X_1 \times \cdots \times X_m}).$$

Letting $n_i = [\dim(X_i)/2]$ for all $1 \leq i \leq m$, it follows that the set $\mathcal{B}$ consisting of all m-fold external products $\beta_1 \times \cdots \times \beta_m$ with $\beta_j \in \{h^i, l_i \mid 0 \leq i \leq n_j\}$ is an $\mathbb{F}_2$-basis of $\mathrm{Ch}(\overline{X_1 \times \cdots \times X_m})$ (of course, one also has the analogous basis of the integral group $\mathrm{CH}(\overline{X_1 \times \cdots \times X_m})$ consisting of external products of elements of the form $H^?$ or $L_?$). We shall henceforth refer to the set $\mathcal{B}$ as "the" *standard basis* of the latter space. If $\alpha \in \mathrm{Ch}(\overline{X_1 \times \cdots \times X_m})$, then there exist unique elements $n_b \in \mathbb{F}_2$ such that $\alpha = \sum_{b \in \mathcal{B}} n_b \cdot b$. If $n_b = 1$ for a given $b \in \mathcal{B}$, then we shall say that α *involves* b. We will now record some basic facts concerning the groups $\mathrm{Ch}(\overline{X_1 \times \ldots \times X_m})$. Note that this material is implicit in [2, Ch. XIII] (and in earlier work of Vishik; cf. [23]).

3.B. **Isotropic reduction.** Suppose that K/k is a field extension over which none of the φ_i are split. For each $1 \leq i \leq m$, let a_i denote the Witt index of $(\varphi_i)_K$, and let Y_i be the (smooth) projective quadric defined by the vanishing of $\big((\varphi_i)_K\big)_{\mathrm{an}}$ over K.

Lemma 3.2. *There are unique group homomorphisms*

$$f\colon \mathrm{Ch}(\overline{Y_1 \times \cdots \times Y_m}) \to \mathrm{Ch}(\overline{X_1 \times \cdots \times X_m}), \quad g\colon \mathrm{Ch}(\overline{X_1 \times \cdots \times X_m}) \to \mathrm{Ch}(\overline{Y_1 \times \cdots \times Y_m})$$

with the following properties:

(1) *The map f (resp. g) raises (resp. lowers) the dimension of homogeneous cycle classes by the factor $a_1 + \cdots + a_m$;*
(2) *Given elements $\alpha_j \in \{h^i, l_i \mid 0 \leq i \leq n_j - a_j\} \subset \mathrm{Ch}(\overline{Y_j})$ $(1 \leq j \leq m)$, we have*

$$f(\alpha_1 \times \cdots \times \alpha_m) = \beta_1 \times \cdots \times \beta_m, \quad \text{where} \quad \beta_j = \begin{cases} h^{i+a_j} & \text{if } \alpha_j = h^i \\ l_{i+a_j} & \text{if } \alpha_j = l_i; \end{cases}$$

(3) *Given elements $\beta_j \in \{h^i, l_i \mid a_j \leq i \leq n_j\} \subset \mathrm{Ch}(\overline{X_j})$ $(1 \leq j \leq m)$, we have*

$$g(\beta_1 \times \cdots \times \beta_m) = \alpha_1 \times \cdots \times \alpha_m, \quad \text{where} \quad \alpha_j = \begin{cases} h^{i-a_j} & \text{if } \beta_j = h^i \\ l_{i-a_j} & \text{if } \beta_j = l_i; \end{cases}$$

(4) *For any field extension L/K, both f and g send L-rational elements to L-rational elements.*

Proof. As discussed above, there are natural identifications of $\mathrm{Ch}(\overline{X_1 \times \cdots \times X_m})$ with $\bigotimes_{i=1}^m \mathrm{Ch}(\overline{X_i})$ and $\mathrm{Ch}(\overline{Y_1 \times \cdots \times Y_m})$ with $\bigotimes_{i=1}^m \mathrm{Ch}(\overline{Y_i})$. Under these identifications, $f = \bigotimes_{i=1}^m (in^{X_i})_*$ and $g = \bigotimes_{i=1}^m (pr^{X_i})_*$ are the desired homomorphisms (in^{X_i} and pr^{X_i} being the mod-2 reductions of the correspondences considered in §2.E above). $\square$

3.C. **Rationality constraints.** We now specialize to the case where $\varphi_2 = \cdots = \varphi_m$. Working under this assumption, let us set $\varphi := \varphi_2 = \cdots = \varphi_m$, $X := X_2 = \cdots = X_m$ and $n := n_2 = \cdots = n_m$.

Lemma 3.3. *Suppose that there exists a k-rational element of $\mathrm{Ch}(\overline{X_1 \times X \times \cdots \times X})$ involving the standard basis element $l_a \times h^{b_1} \times \cdots \times h^{b_{m-1}}$ for some $0 \leq a \leq n_1$ and $0 \leq b_1, \ldots, b_{m-1} \leq n$. Let $b = \max(b_1, \ldots, b_{m-1})$. Then, for any field extension K/k,*

$$\mathfrak{i}_W(\varphi_K) > b \quad \Rightarrow \quad \mathfrak{i}_W((\varphi_1)_K) > a.$$

Proof. Among all k-rational elements of $\mathrm{Ch}(\overline{X_1 \times X \times \cdots \times X})$ involving $l_a \times h^{b_1} \times \cdots \times h^{b_{m-1}}$, let α be one involving the least number of standard basis elements possible (so that α is, in particular, homogeneous). Let K/k be an extension for which $\mathfrak{i}_W(\varphi_K) > b$. To prove that $\mathfrak{i}_W((\varphi_1)_K) > a$, we have to show that $l_a \in \mathrm{Ch}_a(\overline{X_1})$ is K-rational (Lemma

2.3). Since $h^{\dim(X_1)-a} \in \mathrm{Ch}_a(\overline{X_1})$ is K-rational, it suffices to show that there exists a K-rational element $\beta \in \mathrm{Ch}(\overline{X \times \cdots \times X})$ with the property that the push-forward

$$\beta_*\colon \mathrm{Ch}(\overline{X_1 \times X \times \cdots \times X}) \to \mathrm{Ch}(\overline{X_1})$$

(see 2.B above) sends α to $l_a + ch^{\dim(X_1)-a}$ for some $c \in \mathbb{F}_2$. For each (possibly empty) subset I of $\{1,\ldots,m-1\}$, let $\gamma_I \in \mathrm{Ch}(\overline{X_1 \times X \times \cdots \times X})$ be the element obtained from $l_a \times h^{b_1} \times \cdots \times h^{b_{m-1}}$ by replacing the factor h^{b_i} with l_n for each $i \in I$. Let $0 \leq j < m$ be maximal so that α involves an element γ_I with $|I| = j$ (this makes sense by our choice of α). Permuting the factors of $X \times \cdots \times X$ if necessary, we can then assume that α involves

$$\gamma_{\{1,\ldots,j\}} = l_a \times \underbrace{l_n \times \cdots \times l_n}_{j \text{ times}} \times h^{b_{i+1}} \times \cdots \times h^{b_{m-1}}.$$

A quick calculation (using the maximality of j and the minimality of α) now shows that

$$\beta := \underbrace{h^n \times \cdots \times h^n}_{j \text{ times}} \times l_{b_{i+1}} \times \cdots \times l_{b_{m-1}}$$

has the desired property (note that β is K-rational by Lemma 2.3 and our choice of K). $\square$

3.D. **Self-correpondences on quadrics.** We specialize further to the case where $m = 2$ and $\varphi_1 = \varphi_2$. Working under this assumption, let us now set $\varphi := \varphi_1 = \varphi_2$, $X := X_1 = X_2$ and $n := n_1 = n_2$. Let $\mathfrak{j}_1, \ldots, \mathfrak{j}_{h(\varphi)}$ denote the absolute higher Witt indices of φ (see [2, §25]). Letting $d = \dim(X)$, we set $\mathrm{Ch}_{\geq d}(\overline{X \times X}) = \bigoplus_{i \geq d} \mathrm{Ch}_i(\overline{X \times X})$. The first part of the following proposition follows from Lemma 3.3 above. The second part is a variant of a basic result of Vishik ([23, Cor. 4.14]) due to Karpenko. For a proof, see [2, Lem. 73.19] (though formulated in a slightly less general way, the statement of the latter is readily seen to be equivalent to the statement below):

Proposition 3.4. *Assume that φ is anisotropic, and let $\alpha \in \mathrm{Ch}_{\geq d}(\overline{X \times X})$ be a k-rational element. Let $\mathfrak{j}_{r-1} \leq i < \mathfrak{j}_r$ for some $1 \leq r \leq h(\varphi)$ and let $0 \leq j \leq n$.*
  (1) *If α involves the standard basis elements $h^i \times l_j$ or $l_j \times h^i$, then $i \leq j < \mathfrak{j}_r$.*
  (2) *If $i \leq j < \mathfrak{j}_r$, then α involves $h^i \times l_j$ if and only if it involves $l_{\mathfrak{j}_{r-1}+\mathfrak{j}_r-i-1} \times h^{\mathfrak{j}_{r-1}+\mathfrak{j}_r-j-1}$.*

3.E. **Motives of quadrics.** We will write $\mathit{Chow}(k)$ for the category of Chow motives over k with *integral* coefficients (see, e.g., [2, Ch. XII]). If Z is a smooth projective k-variety, we write $M(Z)$ for its motive in $\mathit{Chow}(k)$. In the case where $Z = \mathrm{Spec}(k)$, we simply write $\mathbb{Z}$ instead of $M\big(\mathrm{Spec}(k)\big)$. Given any object M in $\mathit{Chow}(k)$, and any integer i, we write $M\{i\}$ for the ith Tate twist of M in $\mathit{Chow}(k)$. Recall that the Chow group functors on the category of smooth projective k-varieties are extended to $\mathit{Chow}(k)$ by setting $\mathrm{CH}_i(M) := \mathrm{Hom}(\mathbb{Z}\{i\}, M)$ and $\mathrm{CH}^i(M) := \mathrm{Hom}(M, \mathbb{Z}\{i\})$.

Let φ, X, n and d be as in §3.D. For each integer $0 \leq j \leq n$, set $\Lambda(j) = \{i \mid 0 \leq i \leq j\} \coprod \{d - i \mid 0 \leq i \leq j\}$. If Y denotes the projective quadric defined by the anisotropic part of φ, then one has an isomorphism

$$M(X) \cong \left(\bigoplus_{i \in \Lambda(\mathfrak{i}_W(\varphi)-1)} \mathbb{Z}\{i\} \right) \bigoplus M(Y)\{\mathfrak{i}_W(\varphi)\} \tag{3.1}$$

(here $M(Y) = 0$ if φ is split; see [23, Prop. 2.1]). Now, by a result of Vishik (see [23, §§3,4] or [2, Ch. XVII]), any direct summand of $M(X)$ decomposes (in an essentially unique way) into a finite direct sum of indecomposable objects in $\mathit{Chow}(k)$. Since $M(X_{\overline{k}}) \cong \bigoplus_{i \in \Lambda(n)} \mathbb{Z}\{i\}$, it follows that if N is a non-zero direct summand of $M(X)$, then there exists a unique non-empty subset $\Lambda(N) \subseteq \Lambda(n)$ such that $N_{\overline{k}} \cong \bigoplus_{\lambda \in \Lambda(N)} \mathbb{Z}\{\lambda\}$. Moreover, if N and N' are distinct *indecomposable* direct summands of $M(X)$, then it is easy to see that $\Lambda(N)$ and $\Lambda(N')$ are disjoint, and so the complete motivic decomposition of X determines a partition of the set $\Lambda(n)$. We refer to this partition as the *motivic decomposition type* of X. As outlined in [2, Ch. XVII], the determination of this invariant is equivalent to the determination of the k-rational part of the $\mathbb{F}_2$-vector space $\mathrm{Ch}_{\geq d}(\overline{X \times X})$ (see also [4]). For example, Proposition 3.4 (2) above is best interpreted as a statement about the motivic decomposition type of X. We will interchange between these viewpoints frequently. The unique indecomposable direct summand N of $M(X)$ for which $0 \in \Lambda(N)$ (equivalently, for which $\mathrm{CH}^0(N) = \mathbb{Z} \cdot [X]$) is called the *upper motive* of X, and is denoted $U(X)$.

4. On stable birational equivalence of quadratic forms

If σ and τ are anisotropic quadratic forms of dimension ≥ 2 over a field k, then σ and τ are said to be *stably birationally equivalent* if both $\varphi_{k(\psi)}$ and $\psi_{k(\varphi)}$ are isotropic (when $\sigma \not\cong \mathbb{H} \not\cong \tau$, this is easily seen to be equivalent to requiring that the projective quadrics associated to σ and τ are stably birationally isomorphic). We have the following necessary condition for stable birational equivalence of anisotropic forms due to Vishik (see [23, Cor. 4.9], or [2, Thm. 76.5] for a characteristic-free proof):

Proposition 4.1. *If σ and τ are stably birationally equivalent anisotropic quadratic forms of dimension ≥ 2 over a field k, then* $\dim(\sigma) - \mathfrak{i}_1(\sigma) = \dim(\tau) - \mathfrak{i}_1(\tau)$.

Suppose now that φ and φ' are anisotropic quadratic forms of dimension ≥ 2 over a field F, and let X and X' denote their associated (smooth) projective quadrics. Let K/F be a field extension over which neither φ nor φ' is split, and let $\psi = (\varphi_K)_{\mathrm{an}}$ and $\psi' = (\varphi'_K)_{\mathrm{an}}$.

Proposition 4.2. *Suppose, in the above situation, that ψ and ψ' are stably birationally equivalent. Then:*

(1) $\dim(\psi) - \mathfrak{i}_1(\psi) = \dim(\psi') - \mathfrak{i}_1(\psi')$*;*
(2) *Let $d = \dim(\psi) - \mathfrak{i}_1(\psi) = \dim(\psi') - \mathfrak{i}_1(\psi')$. If a and b are integers satisfying*
$$\mathfrak{i}_W(\varphi_K) \leq a < \dim(\varphi) - d - \mathfrak{i}_W(\varphi_K)$$
and
$$\mathfrak{i}_W(\varphi'_K) \leq b < \dim(\varphi') - d - \mathfrak{i}_W(\varphi'_K),$$
then any F-rational element of $\mathrm{Ch}(\overline{X \times X'})$ involving the standard basis element $h^a \times l_b$ also involves $l_{\dim(\varphi)-d-a-1} \times h^{\dim(\varphi')-d-b-1}$.

Proof. Using the map g of Lemma 3.2 (with $k = K$, $m = 2$, $\varphi_1 = \varphi$ and $\varphi_2 = \varphi'$), we reduce to the case where $K = F$. Claim (2) is then that any F-rational element $\alpha \in \mathrm{Ch}(\overline{X \times X'})$ involving $h^a \times l_b$ also involves $l_{\mathfrak{i}_1(\varphi)-a-1} \times h^{\mathfrak{i}_1(\varphi')-b-1}$. Suppose, for the sake of contradiction, that this is not the case. Since φ and φ' are stably birationally equivalent, the Witt index of φ over $F(\varphi')$ is equal to $\mathfrak{i}_1(\varphi)$. In other words, $l_{\mathfrak{i}_1(\varphi)-1} \in \mathrm{Ch}(\overline{X})$ is $F(\varphi')$-rational (Lemma 2.3). By Lemma 3.1, it follows that there exists an F-rational element $\beta \in \mathrm{Ch}(\overline{X' \times X})$ involving the standard basis element $h^0 \times l_{\mathfrak{i}_1(\varphi)-1}$. If we let $\gamma = \beta \cdot (h^b \times h^0)$, then the composition $\gamma \circ \alpha \in \mathrm{Ch}(\overline{X \times X})$ involves $h^a \times l_{\mathfrak{i}_1(\varphi)-1}$ but not $l_{\mathfrak{i}_1(\varphi)-a-1} \times h^0$. Since $\gamma \circ \alpha$ is F-rational, this contradicts Proposition 3.4 (2), and so the lemma follows. □

In order to apply this proposition in the sequel, we will need the following important theorem originally proved by Izhboldin in characteristic not 2 (see [8, Cor. 2.12]) and later in characteristic 2 by Hoffmann and Laghribi ([7, Prop. 4.6]):

Theorem 4.3. *Let σ and τ be anisotropic quadratic forms of dimension ≥ 2 over a field k. Suppose that there exists a positive integer n such that $2^n < \dim(\sigma), \dim(\tau) \leq 2^{n+1}$, and suppose that $\dim(\sigma) - \mathfrak{i}_1(\sigma) = 2^n$. If $\sigma_{k(\tau)}$ is isotropic, then σ and τ are stably birational equivalent, and $\dim(\tau) - \mathfrak{i}_1(\tau) = 2^n$.*

5. On binary direct summands in the motives of quadrics

Henceforth, we will work over a fixed base field F of any characteristic. For the rest of the present section, we also fix an anisotropic quadratic form φ of dimension ≥ 2 over F. We let X denote the (smooth) projective quadric defined by the vanishing of φ, and we let $\mathfrak{j}_1, \ldots, \mathfrak{j}_{h(\varphi)}$ (resp. $\mathfrak{i}_1, \ldots, \mathfrak{i}_{h(\varphi)}$) be the absolute (resp. relative) higher Witt indices of φ (again, see see [2, §25]). The purpose of this section is to prove the following statement:

Proposition 5.1. *Suppose, in the above situation, that there exists an integer $1 \leq r \leq h(\varphi)$ such that $\mathfrak{i}_r > \mathfrak{i}_t$ for all $r < t \leq h(\varphi)$. Then:*

1. *$\dim(\varphi_{r-1}) - \mathfrak{i}_r = 2^n$ for some positive integer n.*
2. *If, in addition, $\mathfrak{j}_{r-1} < \min(\mathfrak{i}_r, 2^{n-1})$, then there exists a motive M in $\mathsf{Chow}(F)$ such that*
   - *$\bigoplus_{i=0}^{\mathfrak{i}_r-1} M\{i\}$ is a direct summand of $M(X)$, and*
   - *$M_{\overline{F}} \cong \mathbb{Z}\{\mathfrak{j}_{r-1}\} \bigoplus \mathbb{Z}\{\mathfrak{j}_{r-1} + 2^{n-1} - 1\}$.*

Remark 5.2. In the language of [23], (2) says that, under the stated assumptions, the rth shell of $M(X)$ consists of binary motives. Note that it has been conjectured by Vishik ([23, Conj. 4.21]) that any binary direct summand in the motive of a smooth projective quadric is isomorphic to a Rost motive (i.e., the upper motive of a Pfister quadric) up to Tate twisting. We do not show in our situation that $M\{-\mathfrak{j}_{r-1}\}$ is the Rost motive of an n-fold Pfister quadric; the existence of M will be enough for our purposes.

Before proving the proposition, we record the following application:

Corollary 5.3. *Let n be a positive integer and let $2^{n-2} < m \leq 2^n$. Suppose, in the above situation, that $2^n + m \leq \dim(\varphi) \leq \min(2^n + 3m, 2^{n+1} + m)$. If there exists a positive integer $1 \leq r \leq h(\varphi)$ such that $\dim(\varphi_{r-1}) = 2^n + m$ and $\dim(\varphi_r) < 2^n$, then $\mathfrak{i}_r = m$ and there exists a motive M in $\mathsf{Chow}(F)$ such that*

- *$\bigoplus_{i=0}^{\mathfrak{i}_r-1} M\{i\}$ is a direct summand of $M(X)$, and*
- *$M_{\overline{F}} \cong \mathbb{Z}\{\mathfrak{j}_{r-1}\} \bigoplus \mathbb{Z}\{\mathfrak{j}_{r-1} + 2^{n-1} - 1\}$.*

Proof. For each $1 \leq t \leq h(\varphi)$, let u_t be the smallest non-negative integer such that $\mathfrak{i}_t \leq 2^{u_t}$. By Karpenko's theorem on the values of the first higher Witt index ([12], [18, Prop. 10.4]), we then have that $\dim(\varphi_{t-1}) - \mathfrak{i}_t \equiv 0 \pmod{2^{u_t}}$. Now, by hypothesis, we have

$$\dim(\varphi_{r-1}) - \mathfrak{i}_r = \dim(\varphi_r) + \mathfrak{i}_r < 2^n + \mathfrak{i}_r,$$

and so $\dim(\varphi_{r-1}) - \mathfrak{i}_r = 2^n - a2^{u_r}$ for some non-negative integer a. Clearly this can only hold if $a = 0$, and so $\mathfrak{i}_r = m$. In particular, $\dim(\varphi_r) = \dim(\varphi_{r-1}) - 2m = 2^n - m$. Since $m > 2^{n-2}$, it follows that $\dim(\varphi_r) < 2^{n-1} + 2^{n-2}$, and so the theorem on the first higher Witt index gives that $\mathfrak{i}_t < 2^{n-2} < m = \mathfrak{i}_r$ for all $r < t \leq h(\varphi)$ (of course, it suffices to use the separation theorem here). Since we also have that

$$\mathfrak{j}_{r-1} = \frac{\dim(\varphi) - (2^n + m)}{2} \leq \frac{\min(m, 2^n)}{2} = \min(2^{n-1}, m) = \min\left(\frac{\dim(\varphi_{r-1}) - \mathfrak{i}_r}{2}, \mathfrak{i}_r\right),$$

the conditions of Proposition 5.1 (2) are satisfied, and so the result follows. $\square$

Remark 5.4. Again, Corollary 5.3 asserts that, under the stated hypotheses, the rth shell of $M(X)$ consists of binary motives. In this case, the article [10] of Kahn provides a hypothetical explanation for this phenomenon that tallies well with Vishik's binary motive conjecture, at least if we assume that $\mathrm{char}(F) \neq 2$ and $m > 2^n/3$. Indeed, let $F = F_0 \subset F_1 \subset \cdots \subset F_{h(\varphi)}$ be the Knebusch splitting tower of φ. If $m > 2^n/3$, then

$$2\mathrm{dim}(\varphi_r) = 2^{n+1} - 2m = (2^n + m) + (2^n - 3m) < 2^n + m = \mathrm{dim}(\varphi_{r-1}),$$

and so Kahn's descent conjecture (see [10, Conj. 2, Thm. 3]) predicts that $\varphi_r \cong \tau_{F_r}$ for some anisotropic form τ over F. Since the form $\pi := \varphi - \tau$ becomes hyperbolic over F_r, and since F_r is a tower of function fields of quadratic forms of dimension $> 2^{s-1}$, it lies in $I^s(F)$ (see, e.g., [20, Cor. 3.2]). But $\mathrm{dim}(\pi) \leq \mathrm{dim}(\varphi) + \mathrm{dim}(\varphi_r) = 2^s + 2\mathfrak{j}_{r-1} < 2^s + 2^{s-1}$, and so a theorem of Vishik ([23, Thm. 6.4]) then implies that π is similar to an s-fold Pfister form. Finally, since the conditions $\mathfrak{i}_W(\pi_K) > 0$ and $\mathfrak{i}_W(\varphi_K) > \mathfrak{j}_{r-1}$ are equivalent for any field extension K/F, it then follows from [23, Thm. 4.15] that the rth shell of $M(X)$ consists of Tate twists of the Rost motive associated to π. Note that this picture has been fully realized in the case where $\mathrm{dim}(\varphi_r) \leq 7$ by results of several authors (in particular, Kahn, Laghribi, Izhboldin and Karpenko). For example, the third shell of the motive of a 15-dimensional anisotropic quadric with absolute splitting pattern $(1, 2, 7, 8)$ consists of twists of a 7-dimensional Rost motive (at least in characteristic $\neq 2$).

We now prove Proposition 5.1, which, translated to the language of cycles, says the following (see [2, Ch. XVII] for details on the translation):

Proposition 5.5. *Suppose, in the above situation, that there exists an integer $1 \leq r \leq h(\varphi)$ such that $\mathfrak{i}_r > \mathfrak{i}_t$ for all $r < t \leq h(\varphi)$. Then:*

1. *$\mathrm{dim}(\varphi_{r-1}) - \mathfrak{i}_r = 2^n$ for some positive integer n.*
2. *If, in addition, $\mathfrak{j}_{r-1} < \min(\mathfrak{i}_r, 2^{n-1})$, then the element*

$$(h^{\mathfrak{j}_{r-1}} \times l_{\mathfrak{j}_r - 1}) + (l_{\mathfrak{j}_r - 1} \times h^{\mathfrak{j}_{r-1}}) \in \mathrm{Ch}(\overline{X \times X})$$

is F-rational.

Proof. Let $F = F_0 \subset F_1 \subset \cdots \subset F_{h(\varphi)}$ be the Knebusch splitting tower of φ. Let Y denote the (smooth) projective quadric over F_{r-1} defined by the vanishing of φ_{r-1}. Since $\mathfrak{i}_r > \mathfrak{i}_t$ for all $r < t \leq h(\varphi)$, it follows from [2, Lem. 73.18] and Proposition 3.4 that

$$(h^0 \times l_{\mathfrak{i}_r - 1}) + (l_{\mathfrak{i}_r - 1} \times h^0) \in \mathrm{Ch}(\overline{Y \times Y})$$

is F_{r-1}-rational (alternatively, the upper motive of Y is binary). Statement (1) then follows from the binary motive theorem of Vishik (see [23, Thm. 4.20] or [2, Cor. 80.8] for a formulation in the language of cycles; note that this result is now also valid in characteristic 2 thanks to [18]). Assume now that $\mathfrak{j}_{r-1} < \min(\mathfrak{i}_r, 2^{n-1})$ and let $\alpha = (h^{\mathfrak{j}_{r-1}} \times l_{\mathfrak{j}_r - 1}) + (l_{\mathfrak{j}_r - 1} \times h^{\mathfrak{j}_{r-1}}) \in \mathrm{Ch}(\overline{X \times X})$. We will show that α is F_j-rational for all $0 \leq j < r$ by descending induction on j. The case where $j = r - 1$ follows from Lemma 3.2 (2) and the fact that $h^0 \times l_{\mathfrak{i}_r - 1} + l_{\mathfrak{i}_r - 1} \times h^0 \in \mathrm{Ch}(\overline{Y \times Y})$ is F_{r-1}-rational. Suppose now that $0 \leq j \leq r - 2$. Assuming that α is F_{j+1}-rational, we need to show that α is F_j-rational. By another application of Lemma 3.2, we may assume that $j = 0$. Let $\mathcal{B}$ denote the standard $\mathbb{F}_2$-basis of $\mathrm{Ch}(\overline{X \times X \times X})$ discussed in §3 above. By Lemma 3.1 and the induction hypothesis, there exists an F-rational element $\beta \in \mathrm{Ch}(\overline{X \times X \times X})$ that involves the terms $h^0 \times h^{\mathfrak{j}_{r-1}} \times l_{\mathfrak{j}_r - 1}$ and $h^0 \times l_{\mathfrak{j}_r - 1} \times h^{\mathfrak{j}_{r-1}}$ but no other element of $\mathcal{B}$ having the form $h^0 \times ? \times ?$. Among all such elements β, let us choose one that involves the least number of elements of $\mathcal{B}$ as possible. Then β is homogeneous of dimension $2d + \mathfrak{i}_r - 1$, where $d = \dim X$. Note that since $r \geq 2$, our standing hypotheses imply that $\mathfrak{i}_r > 1$.

Lemma 5.6. *In the above situation, we have that*

$$\beta = (h^0\times\alpha)+u(h^{\mathfrak{j}_{r-1}}\times h^0\times l_{\mathfrak{j}_r-1}+l_{\mathfrak{j}_r-1}\times h^0\times h^{\mathfrak{j}_{r-1}})+v(h^{\mathfrak{j}_{r-1}}\times l_{\mathfrak{j}_r-1}\times h^0+l_{\mathfrak{j}_r-1}\times h^{\mathfrak{j}_{r-1}}\times h^0)$$

for some $u, v \in \mathbb{F}_2$.

Proof. For dimension reasons, β involves no standard basis elements of the form $l_? \times l_? \times l_?$, $l_? \times l_? \times h^?$, $l_? \times h^? \times l_?$ or $h^? \times l_? \times l_?$. Suppose now that β involves a standard basis element of the form $l_a \times h^b \times h^c$ for some $0 \leq a, b, c \leq n = [d/2]$. Let $i = \max(b, c)$, and let $1 \leq t \leq h(\varphi)$ be such that $\mathfrak{j}_{t-1} \leq i < \mathfrak{j}_t$. By Lemma 3.3, it then follows that $\mathfrak{j}_{t-1} \leq a < \mathfrak{j}_t$, and so $a - (b + c) \leq \mathfrak{i}_t - 1$. On the other hand, we have $a - (b + c) = \mathfrak{i}_r - 1$, and so our standing assumption on the splitting pattern of φ implies that $t = r$, $a = \mathfrak{j}_{r-1}$ and either $(b, c) = (0, \mathfrak{j}_r - 1)$ or $(b, c) = (\mathfrak{j}_r - 1, 0)$. By symmetry, the same conclusion stands for all standard basis elements of the form $h^b \times l_a \times h^c$ or $h^b \times h^c \times l_a$ involved in β. We therefore have that

$$\beta = (h^0\times\alpha)+u(h^{\mathfrak{j}_{r-1}}\times h^0\times l_{\mathfrak{j}_r-1})+u'(l_{\mathfrak{j}_r-1}\times h^0\times h^{\mathfrak{j}_{r-1}})+v(h^{\mathfrak{j}_{r-1}}\times l_{\mathfrak{j}_r-1}\times h^0)+v'(l_{\mathfrak{j}_r-1}\times h^{\mathfrak{j}_{r-1}}\times h^0)$$

for some $u, u', v, v' \in \mathbb{F}_2$. It remains to show that $u = u'$ and $v = v'$. To see this, let $\mu = h^0 \times l_0 \times h^0 \in \mathrm{Ch}(\overline{X \times X \times X})$. Then μ is F_1-rational, and pushing $\mu \cdot \beta$ forward via the projection $X \times X \times X \to X \times X$ to the first and third factors, we see that

$$u(h^{\mathfrak{j}_{r-1}} \times l_{\mathfrak{j}_r-1}) + u'(l_{\mathfrak{j}_r-1} \times h^{\mathfrak{j}_{r-1}}) \in \mathrm{Ch}(\overline{X \times X})$$

is also F_1-rational. Let Y now denote the projective quadric defined by the vanishing of φ_1 over F_1 (as opposed to φ_{r-1}). Applying the homomorphism g of Lemma 3.2 (with $k = F_1$, $m = 2$ and $X = X_1 = X_2$), we get that

$$u(h^{\mathfrak{j}'_{r-2}} \times l_{\mathfrak{j}'_{r-1}-1}) + u'(l_{\mathfrak{j}'_{r-1}-1} \times h^{\mathfrak{j}'_{r-2}}) \in \mathrm{Ch}(\overline{Y \times Y})$$

is F_1-rational, where $\mathfrak{j}'_1, \mathfrak{j}'_2, \ldots, \mathfrak{j}'_{h(\varphi)-1}$ denote the absolute higher Witt indices of φ_1 (i.e., $\mathfrak{j}'_{t-1} = \mathfrak{j}_t - \mathfrak{i}_1$). Since $r \geq 2$, it then follows from Proposition 3.4 (2) that $u = u'$. The equality $v = v'$ is shown in a similar way (in the above argument, replace μ by $h^0 \times h^0 \times l_0$ and then use push-forward onto the product of the first two factors of $X \times X \times X$). $\square$

Lemma 5.7. *In the above situation,*

$$\gamma = u(h^{2\mathfrak{j}_{r-1}} \times l_{\mathfrak{j}_r-1} + l_{\mathfrak{i}_r-1} \times h^{\mathfrak{j}_{r-1}}) + v(h^{\mathfrak{j}_{r-1}} \times l_{\mathfrak{i}_r-1} + l_{\mathfrak{j}_r-1} \times h^{2\mathfrak{j}_{r-1}}) \in \mathrm{Ch}(\overline{X \times X})$$

is F-rational.

Proof. The proof uses a construction of Karpenko (cf. [13] or [2, §81]). To simplify the notation, let $a = \mathfrak{j}_{r-1}$ and $b = \mathfrak{j}_r - 1$. Our standing assumption is that $a < \min(b - a + 1, 2^{n-1})$. Let $a \leq j \leq 2a$. Then $j \leq b$, and, by Lemma 2.1, we have

$$S^j(l_b) = \binom{2^n + a}{j} l_{b-j},$$

where S^j denotes the jth Steenrod operation of cohomological type ([1, 18]). Since $a < 2^{n-1}$, however, the binomial coefficient $\binom{2^n+a}{j}$ is odd only in the case where $j = a$ (see, e.g., [22, Lem. 3.4.2]). Since $S^j(h^a) = 0$ for all $j > a$, and since $S^j(h^0) = 0$ for all $j > 0$, it then follows from Lemma 5.6 and the external product formula for Steenrod operations ([2, Thm. 61.14]) that

$$\begin{aligned} S^{2a}(\beta) = {} & (h^0 \times h^{2a} \times l_{b-a} + h^0 \times l_{b-a} \times h^{2a}) + u(h^{2a} \times h^0 \times l_{b-a} + l_{b-a} \times h^0 \times h^{2a}) \\ & + v(h^{2a} \times l_{b-a} \times h^0 + l_{b-a} \times h^{2a} \times h^0). \end{aligned}$$

In particular, the element

$$\eta := S^{2a}(\beta)\cdot(h^0\times h^0\times h^{b-2a}) = (h^0\times h^{2a}\times l_a + h^0\times l_{b-a}\times h^b) + u(h^{2a}\times h^0\times l_a + l_{b-a}\times h^0\times h^b) + v(h^{2a}\times l_{b-a}\times h^{b-2a} + l_{b-a}\times h^{2a}\times h^{b-2a})$$

is F-rational. Consider now the morphisms

$$t_{12}\colon X\times X\times X\to X\times X\times X \quad \text{and} \quad \delta_{X\times X}\colon X\times X\to X\times X\times X\times X$$

defined by the assignments $(x,y,z)\mapsto(y,x,z)$ and $(x,y)\mapsto(x,y,x,y)$, respectively. Viewing η as a geometric correspondence from $X\times X$ to X and $t_{12}^*(\beta)$ as a geometric correspondence in the other direction, we compute that

$$(\delta_{X\times X})^*(t_{12}^*(\beta)\circ\eta) = u(h^{2a}\times l_{b-1} + l_{b-a}\times h^a) + v(h^a\times l_{b-a} + l_a\times h^{2a}).$$

This proves the lemma, since both t_{12}^* and $(\delta_{X\times X})^*$ preserve F-rationality. $\square$

Now, to complete the proof of 5.5, it suffices by [2, Prop. 73.23] to show there exists a non-zero F-rational cycle in $\mathrm{Ch}_{\geq d}(X\times X)$ that does not involve any standard basis elements of the form $h^?\times l_i$ or $l_i\times h^?$ with $i\notin[\mathfrak{j}_{r-1},\mathfrak{j}_r)$ (or, in another language, that there is a direct summand N of $M(X)$ with the property that $N_{\overline{k}}$ does not involve Tate motives lying outside the rth shell of $M(X)$). To this end, consider the morphism

$$\delta_X\times\mathrm{id}\colon X\times X\to X\times X\times X, \quad (x,y)\mapsto(x,x,y).$$

By Lemma 5.6, we have

$$(\delta_X\times\mathrm{id})^*(\beta) = (1+u)\alpha.$$

Since $(\delta_X\times\mathrm{id})^*$ preserves F-rationality, it follows that

$$(1+u)(h^{\mathfrak{j}_{r-1}}\times h^0)(\alpha) = (1+u)(h^{2\mathfrak{j}_{r-1}}\times l_{\mathfrak{j}_r-1} + l_{\mathfrak{i}_r-1}\times h^{\mathfrak{j}_{r-1}})$$

is F-rational. Adding the element γ of Lemma 5.7 to this cycle class, we then get that

$$(h^{2\mathfrak{j}_{r-1}}\times l_{\mathfrak{j}_r-1} + l_{\mathfrak{i}_r-1}\times h^{\mathfrak{j}_{r-1}}) + v(h^{\mathfrak{j}_{r-1}}\times l_{\mathfrak{i}_r-1} + l_{\mathfrak{j}_r-1}\times h^{2\mathfrak{j}_{r-1}}) \in \mathrm{Ch}(\overline{X\times X})$$

is F-rational. Since $0<\mathfrak{j}_{r-1}<\mathfrak{i}_r=\mathfrak{j}_r-\mathfrak{j}_{r-1}$, this proves what we want. $\square$

6. Some special cases of the main conjecture

Following §1, we now let p and q be anisotropic quadratic forms of dimension ≥ 2 over F, P and Q their associated (smooth) projective F-quadrics, and $k=\dim((q_{F(p)})_{\mathrm{an}}) = \dim(q)-2\mathfrak{i}_W(q_{F(p)})$. Let s be the unique non-negative integer satisfying $2^s<\dim(p)\leq 2^{s+1}$. The aim of this section is to note the following proposition (to be used in the sequel):

Proposition 6.1. *Suppose, in the above situation, that either of the following conditions holds:*

(1) *p becomes isotropic over any field extension K/F for which $\dim((q_K)_{\mathrm{an}})\leq k$;*
(2) *$\dim(q)\leq 2^{s+2}+2^{s+1}+k$.*

Then the statement of Conjecture 1.1 holds for the pair (p,q), i.e., $\dim(q)=a2^{s+1}+\epsilon$ for some non-negative integer a and some integer $-k\leq\epsilon\leq k$.

Note that case (1) of the proposition was essentially treated in [20, §5.3]. We will recall the details here since the main result of the previous section now allows us to incorporate case (2) into the argument (moreover, the exposition in [20] contains a minor error).

Proof. We may assume that $k\leq 2^s-2$, since otherwise the statement of Conjecture 1.1 holds trivially. Since the statement also holds trivially when $2^{s+2}+2^{s+1}-k\leq\dim(q)\leq 2^{s+2}+2^{s+1}+k$, (2) may be replaced by

(2') $\dim(q) < 2^{s+2} + 2^{s+1} - k$.

Now, let $F = F_0 \subset F_1 \subset \cdots \subset F_{h(q)}$ be the Knebusch splitting tower of q, and let $0 \leq r \leq h(q)$ be the unique integer for which $q_r = (q_{F_r})_{\mathrm{an}}$ has dimension k. We proceed by induction on r (over all field extensions of F). If $r = 0$, then there is nothing to prove, so assume that $r \geq 1$. Since q_{r-1} becomes isotropic over $F_{r-1}(p)$, and since $\dim(p) > 2^s$, the separation theorem implies that $\dim(q_{r-1}) > 2^s$. Since $\dim(q_r) = k < 2^s$, Karpenko's theorem on the values of the first higher Witt index ([12], [18, Prop. 10.4]) then implies that $\dim(q_{r-1}) = 2^{n+1} - k$ for some integer $n \geq s$. In particular, the statement of Conjecture 1.1 holds for the pair (p, q) when $r = 1$. Assume now that $r \geq 2$. Applying the induction hypothesis to q_1 (over F_1), we get that $\dim(q_1) = a_1 2^{s+1} + \epsilon_1$ for some *positive* integer a_1 and some $-k \leq \epsilon_1 \leq k$. If $\mathfrak{i}_1(q) \leq \frac{k-\epsilon_1}{2}$, then $a_1 2^{s+1} - k \leq \dim(q) \leq a_1 2^{s+1} + k$ and so the desired assertion holds with $a = a_1$. We can therefore assume that $\mathfrak{i}_1(q) > \frac{k-\epsilon_1}{2}$. Let u be the smallest non-negative integer satisfying the inequality $\mathfrak{i}_1(q) \leq 2^u$. By another application of the theorem on the first higher Witt index, we then have that

$$a_1 2^{s+1} + \epsilon_1 + \mathfrak{i}_1(q) = \dim(q) - \mathfrak{i}_1(q) \equiv 0 \pmod{2^u}.$$

We claim that $u \geq s$. Note that this will complete the proof of the proposition, since it implies that $\mathfrak{i}_1(q) \equiv -\epsilon_1 \pmod{2^s}$, and hence

$$\dim(q) = a_1 2^{s+1} + \epsilon_1 + 2\mathfrak{i}_1(q) \equiv -\epsilon_1 \pmod{2^{s+1}},$$

i.e., the desired assertion holds with $\epsilon = -\epsilon_1$. To prove the claim, suppose to the contrary that $u < s$, so that $\mathfrak{i}_1(q) = \mu 2^u - \epsilon_1$ for some integer μ. Since $\mathfrak{i}_1(q) > \frac{k-\epsilon_1}{2}$, we have

$$\mu 2^u = \mathfrak{i}_1(q) + \epsilon_1 > \left(\frac{k - \epsilon_1}{2}\right) + \epsilon_1 = \frac{k + \epsilon_1}{2} \geq 0.$$

At the same time, we also have

$$\mu 2^u = \mathfrak{i}_1(q) + \epsilon_1 \leq 2^u + k < 2^u + 2^s,$$

and so $0 < \mu 2^u \leq 2^s$. If $\mu 2^u = 2^s$, then we again have that $\mathfrak{i}_1(q) \equiv -\epsilon_1 \pmod{2^s}$, and the result follows as above. It will therefore be enough to show that we cannot have $0 < \mu 2^u < 2^s$. Suppose otherwise. Since $\dim(q) - \mathfrak{i}_1(q) = a_1 2^{s+1} + \epsilon_1 + \mathfrak{i}_1(q) = a_1 2^{s+1} + \mu 2^u$, it then follows from Vishik's theorem on excellent connections in the motives of smooth projective quadrics (more precisely, from [25, Thm. 2.1], which is now also valid in characteristic 2 thanks to [18]) that the Tate motive $\mathbb{Z}\{a_1 2^s\}$ is isomorphic to a direct summand of $U(Q)_{\overline{F}}$, where $U(Q)$ denotes the upper motive of Q (see §3.E). Let $j = \mathfrak{i}_W(q_{F(p)}) - 1 - a_1 2^s$. Then

$$j = \left(\frac{\dim(q) - k}{2}\right) - 1 - a_1 2^s = \mathfrak{i}_1(q) - 1 - \left(\frac{k - \epsilon_1}{2}\right),$$

and so $0 \leq j < \mathfrak{i}_1(q)$ (recall that $\mathfrak{i}_1(q) > (k - \epsilon_1)/2$). By [23, Thm. 4.13] ([2, §73] in any characteristic), it follows that $U(Q)\{j\}$ is isomorphic to a direct summand of $M(Q)$. Now, by the preceding discussion, the Tate motive $\mathbb{Z}\{a_1 2^s + j\} = \mathbb{Z}\{\mathfrak{i}_W(q_{F(p)}) - 1\}$ is isomorphic to a direct summand of $U(Q)\{j\}_{\overline{F}}$. To obtain the desired contradiction, it therefore suffices to show that $M(Q)$ admits an indecomposable direct summand N with the property that $\mathbb{Z}\{\mathfrak{i}_W(q_{F(p)}) - 1\}$ is isomorphic to a direct summand of $N_{\overline{F}}$, but $\mathbb{Z}\{j\}$ is not. Suppose first that we are in case (1), i.e., that p becomes isotropic over any field extension K/F for which $\dim((q_K)_{\mathrm{an}}) \leq k$. In this case, it follows from [23, Thm. 4.15] (the proof of which is valid in any characteristic) that $M(Q)$ admits a direct summand isomorphic to $U(P)\{\mathfrak{i}_W(q_{F(p)}) - 1\}$. Now the Tate motive $\mathbb{Z}\{\mathfrak{i}_W(q_{F(p)}) - 1\}$ is isomorphic to a direct summand of $U(P)\{\mathfrak{i}_W(q_{F(p)}) - 1\}_{\overline{F}}$ by definition. At the same time, this is not true of $\mathbb{Z}\{j\}$, since $j < \mathfrak{i}_W(q_{F(p)}) - 1 - a_1 2^s < \mathfrak{i}_W(q_{F(p)}) - 1$ (remember that

$a_1 > 0$). Thus, the unique direct summand N of $M(Q)$ satisfying $N \cong U(P)\{\mathfrak{i}_W(q_{F(p)})-1\}$ has the desired properties in this case. Suppose now that we are in case (2'), i.e., that $\dim(q) < 2^{s+2} + 2^{s+1} - k$. Recall that $\dim(q_{r-1}) = 2^{n+1} - k$ for some integer $n \geq s$. If $\dim(q_{r-1}) = 2^{s+1} - k$ (i.e., $n = s$), then it follows from Izhboldin's Theorem 4.3 that q_{r-1} is stably birational equivalent to $p_{F_{r-1}}$ (note that the latter form is anisotropic, since q_{r-1} becomes isotropic over its function field). By the specialization results of Knebusch ([17, §3]), it then follows that condition (1) holds, and so may conclude as before. Assume now that $\dim(q_{r-1}) = 2^{n+1} - k$ for some integer $n > s$. We then have that

$$\mathfrak{i}_r(q) = \mathfrak{i}_1(q_{r-1}) = 2^n - k \geq 2^{s+1} - k > k > \dim(q_t) > \mathfrak{i}_t(q)$$

for all $r < t \leq h(q)$. At the same time, our assumption on $\dim(q)$ gives that

$$\mathfrak{j}_{r-1}(q) = \frac{\dim(q) - (2^{n+1} - k)}{2} < \frac{(2^{s+2} + 2^{s+1} - k) - (2^{s+2} - k)}{2} = 2^s \leq \min(\mathfrak{i}_r(q), 2^{n-1}),$$

and so Proposition 5.1 tells us that the rth shell of $M(Q)$ consists of binary motives, i.e., there is a motive M in $\mathit{Chow}(F)$ such that

- $\bigoplus_{i=0}^{\mathfrak{i}_r(q)-1} M\{i\}$ is a direct summand of $M(Q)$, and
- $M_{\overline{F}} \cong \mathbb{Z}\{\mathfrak{j}_{r-1}(q)\} \bigoplus \mathbb{Z}\{\mathfrak{j}_{r-1}(q) - 2^n - 1\}$.

Since $\mathfrak{i}_W(q_{F(p)}) = \mathfrak{j}_r(q)$, the summand $N := M\{\mathfrak{i}_r(q) - 1\}$ then has the desired properties (again, we are using here that $j < \mathfrak{i}_W(q_{F(p)}) - 1$). This completes the induction step and the proof of the proposition. $\square$

Remarks 6.2. (1) Let $F = F_0 \subset F_1 \subset \cdots \subset F_{h(q)}$ be the Knebusch splitting tower of q, and let $0 \leq r \leq h(q)$ be the unique integer for which $q_r = (q_{F_r})_{\text{an}}$ has dimension k. As implicitly observed in the proof of the proposition, the following are then equivalent:
- p becomes isotropic over any field extension K/F for which $\dim((q_K)_{\text{an}}) \leq k$;
- p becomes isotropic over F_r;
- There exists a field extension of F over which the dimension of the anisotropic part of q lies in the interval $(2^s, 2^{s+1}]$;
- $r \geq 1$ and $\dim(q_{r-1}) = 2^{s+1} - k$.

(2) Note that the inductive argument used here yields a stronger result: Assume that $k < 2^s$. Then, in either of the two cases under consideration, the statement of [20, Conjecture 5.1] holds for the pair (p, q). More precisely, let $0 \leq r \leq h(q)$ be the unique non-negative integer for which $\dim(q_r) = k$. Then, for each $0 \leq t < r$, there exist non-negative integers a_t, b_t, and integers $-k \leq \epsilon_t \leq k$ such that:
- $\dim(q_t) = a_t 2^{s+1} + \epsilon_t$; and
- with one possible exception, either $\mathfrak{i}_{t+1}(q) \leq \frac{k+\epsilon_t}{2}$ or $\mathfrak{i}_{t+1}(q) = b_t 2^{s+1} + \epsilon_t$. The possible exception is where $\dim(q_t) = 2^{s+1} - k$, in which case $t = r - 1$ and $\mathfrak{i}_{t+1}(q) = 2^s - k$

We omit the details, but the reader is referred to [20, §5] for further information.

7. A technical proposition

In this section we state and prove a technical proposition that will serve as the basic tool in the proof of our main theorem. We continue with the set up of the previous section. We now also let d_P and d_Q denote the dimensions of P and Q respectively, and set $n_P = [d_P/2]$ and $n_Q = [d_Q/2]$. Finally, we fix a non-negative integer $n_Q - n_P < m \leq n_Q$ (recall that s denotes the unique non-negative integer for which $2^s < \dim(p) \leq 2^{s+1}$).

Proposition 7.1. *Assume, in the above situation, that the element $h^0 \times l_m \in \mathrm{Ch}_{d_P+m}(\overline{P \times Q})$ is F-rational and let $0 \leq j \leq m$ be an integer satisfying $j > d_Q - m - 2^{s+1} + 2$. Suppose that there exists a field extension K/F with the following properties:*

(1) $\mathfrak{i}_W(q_K) > m$;
(2) $\mathfrak{i}_W(p_K) \leq d_P - d_Q + j + m + 1$;
(3) *If $a \leq n_Q - m$ and $\eta \in \mathrm{Ch}^a(P_K)$ is the mod-2 reduction of a torsion element of $\mathrm{CH}^a(U(P)_K)$, then $\deg_{d_P-a-b}(\mu) \equiv 0 \pmod 4$ for any pair (b, μ) consisting of an integer $b \geq 0$ and an integral representative $\mu \in \mathrm{CH}^{a+b}(P_K)$ of $S^b(\eta) \in \mathrm{Ch}^{a+b}(P_K)$.*

Then the binomial coefficient $\binom{d_Q-m+1}{j}$ is even.

Remark 7.2. We remind the reader that $U(P)$ denotes here the upper motive of P (§3.E). We also remark that when the proposition is applied in the sequel, the field K will have the property that p_K is anisotropic. In particular, replacing (2) with the more visually appealing condition that p_K is anisotropic will not affect anything that follows.

Proof. Since $m > n_Q - n_P$, we have that $d_P + m > n_P + n_Q$, and so $2\mathrm{CH}_{d_P+m}(\overline{P \times Q})$ consists of F-rational elements (see §3.A). Since the mod-2 cycle class $h^0 \times l_m$ is F-rational, it follows that there exists an integral class $B \in \mathrm{CH}_{d_P+m}(P \times Q)$ such that $\overline{B} = H^0 \times L_m$. Let $\nu \in \mathrm{CH}_{d_P}(P \times P)$ be the projector defining the upper motive $U(P)$. Replacing B with $B \circ \nu$ if necessary, we can assume that $B \circ \nu = \nu$ (note that $\overline{B \circ \nu} = H^0 \times L_m$). Let $\beta \in \mathrm{Ch}_{d_P+m}(P \times Q)$ denote the mod-2 reduction of B (so that $\overline{\beta} = h^0 \times l_m$). The proof now begins by observing an identity implicit in [14, Proof of Thm. 1.2] (having its origin in [24]). Let P' be a $(2^s - 1)$-dimensional subquadric of P, let $\pi'_Q \colon P' \times Q \to Q$ be the canonical projection, and let ι denote the natural embedding of $P' \times Q$ into $P \times Q$. Since $(\pi'_Q)_*\big(\iota^*(\beta)\big) \in \mathrm{Ch}(Q)$ is homogeneous of codimension $d_Q - m - \dim(P') = d_Q - m - 2^s + 1$, it is annihilated by the Steenrod operation S^{2^s-1+a} for any $a > d_Q - m - 2^{s+1} + 2$ ([2, Thm. 61.13]). In particular, this holds for $a = j$. By [2, Prop. 61.10], it follows that

$$0 = S^{2^s-1+j}\Big((\pi'_Q)_*\big(\iota^*(\beta)\big)\Big) = \sum_{l=0}^{2^s-1} (\pi'_Q)_*\Big(\big(c_l(-T_{P'}) \times h^0\big) \cdot S^{2^s-1+j-l}\big(\iota^*(\beta)\big)\Big)$$

(here $-T_{P'}$ denotes the virtual normal bundle of P' and $c_l(-T_{P'})$ its lth Chern class modulo 2). Using the fact that Steenrod operations commute with pull-backs ([2, Thm. 61.9]) together with the projection formula for proper push-forwards ([2, Prop. 56.9]) we note that the sum appearing on the right side of the preceding expression equals

$$\sum_{l=0}^{2^s-1} (\pi_Q)_*\big(\iota_*(c_l(-T_{P'}) \times h^0) \cdot S^{2^s-1+j-l}(\beta)\big),$$

where $\pi_Q = \pi'_Q \circ \iota$ is the canonical projection from $P \times Q$ to Q. Now, since $\dim(P')$ is one less than a power of 2, we have $c_l(-T_{P'}) = h^l$ for all $l \geq 0$ (see [2, Lem. 78.1]). As $\iota^*(h^l \times h^0) = h^{l+d_P-\dim(P')} \times h^0 = h^{l+d_P-2^s+1} \times h^0$, we deduce from the above that

$$\sum_{l=0}^{2^s-1} \pi_*\big((h^{l+d_P-2^s+1} \times h^0) \cdot S^{2^s-1+j-l}(\beta)\big) = 0$$

in $\mathrm{Ch}(Q)$ (the identity implicit in [14, Proof of Thm. 1.2]). Relabelling $2^s - 1 + j - l$ as i and multiplying through by $h^{m-j} \in \mathrm{Ch}(Q)$, we arrive at the identity

$$\sum_{i=j}^{d_Q-m} h^{m-j} \cdot \pi_*\big((h^{d_P+j-i} \times h^0) \cdot S^i(\beta)\big) = 0$$

in $\mathrm{Ch}_0(Q)$ (again, we have also used here that β is homogeneous of codimension $d_Q - m$, so that $S^i(\beta) = 0$ for all $i > d_Q - m$). Using the projection formula, this becomes

$$\sum_{i=j}^{d_Q-m} (\pi_Q)_*\big((h^{d_P+j-i} \times h^{m-j}) \cdot S^i(\beta)\big) = 0.$$

For each i, let us now fix an integral representative $s_i \in \mathrm{CH}_{d_P+m-i}(P \times Q)$ of the mod-2 class $S^i(\beta)$, specifically taking s_{d_Q-m} to be B^2, where B is the integral lift of β fixed at the beginning of the proof (see [2, Thm. 61.13]). The preceding identity then tells us that

$$\sum_{i=j}^{d_Q-m} (\pi_Q)_*\big((H^{d_P+j-i} \times H^{m-j}) \cdot s_i\big) \in 2\mathrm{CH}_0(Q).$$

Since q is anisotropic, Springer's theorem (see Lemma 2.3) now gives that

$$\sum_{i=j}^{d_Q-m} \deg\Big((\pi_Q)_*\big((H^{d_P+j-i} \times H^{m-j}) \cdot s_i\big)\Big) \equiv 0 \pmod 4.$$

By functoriality of push-forwards, we deduce that

$$\sum_{i=j}^{d_Q-m} \deg\Big((\pi_P)_*\big((H^{d_P+j-i} \times H^{m-j}) \cdot s_i\big)\Big) \equiv 0 \pmod 4,$$

where π_P denotes the canonical projection from $P \times Q$ to Q. By another application of the projection formula, this becomes

$$\sum_{i=j}^{d_Q-m} \deg\Big(H^{d_P+j-i} \cdot (\pi_P)_*\big((H^0 \times H^{m-j}) \cdot s_i\big)\Big) \equiv 0 \pmod 4,$$

or

$$\sum_{i=j}^{d_Q-m} \deg_{d_P+j-i}\Big((\pi_P)_*\big((H^0 \times H^{m-j}) \cdot s_i\big)\Big) \equiv 0 \pmod 4$$

(Lemma 2.2). In particular, in order to prove the proposition, it will be enough to show that

$$\deg_{d_P+j-i}\Big((\pi_P)_*\big((H^0 \times H^{m-j}) \cdot s_i\big)\Big) \equiv \begin{cases} 2\binom{d_Q-m+1}{j} \pmod 4 & \text{if } i = j \\ 0 \pmod 4 & \text{otherwise.} \end{cases} \tag{7.1}$$

Since $\deg_{d_P+j-i}$ is insensitive to scalar extension, we are, with one caveat, free to replace the field F with the field K given in the statement of the proposition. The caveat is that $U(P)_K$ need not be the upper motive of P_K. To avoid any confusion, let us now write N instead of $U(P)$ (so that N_K is the direct summand of $M(P_K)$ defined by the projector ν_K). Replacing F by K, we can then assume that the following hold:

(i) $\mathfrak{i}_W(q) > m$;
(ii) $\mathfrak{i}_W(p) \leq d_P - d_Q + j + m + 1$; and
(iii) If $a \leq n_Q - m$ and $\eta \in \mathrm{Ch}^a(P)$ is the mod-2 reduction of a torsion element of $\mathrm{CH}^a(N)$, then $\deg_{d_P-a-b}(\mu) \equiv 0 \pmod 4$ for any pair (b, μ) consisting of an integer $b \geq 0$ and an integral representative $\mu \in \mathrm{CH}^{a+b}(P)$ of $S^b(\eta) \in \mathrm{Ch}^{a+b}(P)$.

Let us first show that (7.1) holds for $i = d_Q - m$. Since $s_{d_Q-m} = B^2$, we have

$$\overline{s_{d_Q-m}} = \overline{B^2} = \overline{B}^2 = (H^0 \times L_m)^2 = H^0 \times L_m^2$$

in $\mathrm{CH}(\overline{P \times Q})$. Now, since $j \leq m$, we have that $j = d_Q - m$ if and only if $m = n_Q$ (in which case d_Q is even and $j = n_Q$). If $m \neq n_Q$, then $L_m^2 = 0$, and so the degree of interest is 0. If $m = n_Q$, on the other hand, then this degree equals

$$\deg_{d_P}\Big((\pi_P)_*\big(H^0 \times L_{n_Q}^2\big)\Big) = 2\deg(L_{n_Q}^2).$$

Since we also have $j = n_Q$ in this case, it then only remains to check that $\deg(L_{n_Q}^2) \equiv \binom{d_Q-n_Q+1}{n_Q} \pmod 2$. But

$$L_{n_Q}^2 = \begin{cases} L_0 & \text{if } d_Q \text{ is divisible by } 4 \\ 0 & \text{otherwise,} \end{cases}$$

([2, Ex. 68.3]) and so the claim holds.

Let us now assume that $j < d_Q - m$ and fix an integer $j \leq i < d_Q - m$. By (i) and Lemma 2.3, the class $L_m \in \mathrm{CH}_m(\overline{Q})$ descends to a unique element of $\mathrm{CH}_m(Q)$ that we denote in the same way. Since $\overline{B} = H^0 \times L_m$, it follows that $\beta = (h^0 \times l_m) + \gamma$, where $\gamma \in \mathrm{Ch}_{d_P+m}(P \times Q)$ is the mod-2 reduction of the *torsion* integral class $B - (H^0 \times L_m) \in \mathrm{CH}_{d_P+m}(P \times Q)$. For each $l \geq 0$, let $t_l \in \mathrm{CH}_{d_P+m-l}(P \times Q)$ be an integral representative of $S^l(\gamma) \in \mathrm{Ch}_{d_P+m-l}(P \times Q)$, specifically setting $t_0 := B - (H^0 \times L_m)$. Since

$$S^i(h^0 \times l_m) = \binom{d_Q - m + 1}{i}(h^0 \times l_{m-i})$$

$\big($Lemma 2.1 together with the external product formula for Steenrod operations ([2, Thm. 61.14])$\big)$, it follows that

$$s_i \equiv \binom{d_Q - m + 1}{i}(H^0 \times L_{m-i}) + t_i \pmod 2.$$

Now, by (ii) we have

$$\mathfrak{i}_W(p) \leq d_P - d_Q + j + m + 1 = d_P + j - (d_Q - m - 1) \leq d_P + j - i,$$

and so the image of $\deg_{d_P+j-i} \colon \mathrm{CH}_{d_P+j-i}(P) \to \mathbb{Z}$ equals $2\mathbb{Z}$ by Lemma 2.3. Thus, modulo 4, the left side of (7.1) is congruent to

$$\deg_{d_P+j-i}\left((\pi_P)_*\left(\binom{d_Q - m + 1}{i}(H^0 \times L_{j-i}) + (H^0 \times H^{m-j}) \cdot t_i\right)\right)$$

which is readily seen to equal

$$2\delta_{ij}\binom{d_Q - m + 1}{i} + \deg_{d_P+j-i}\Big((\pi_P)_*\big((H^0 \times H^{m-j}) \cdot t_i\big)\Big).$$

(δ_{ij} being the Kronecker delta). To show that (7.1) holds (for this i), it now remains to show that

$$\deg_{d_P+j-i}\big((\pi_P)_*\big((H^0 \times H^{m-j}) \cdot t_i\big)\big) \equiv 0 \pmod 4.$$

We will do this by proving that

$$\deg_{d_P+j-i}\Big((\pi_P)_*\big((H^0 \times H^{m-j+i-l}) \cdot t_l\big)\Big) \equiv 0 \pmod 4 \tag{7.2}$$

for all $0 \leq l \leq i$, arguing by induction on l. Since t_0 is torsion, the assertion holds when $l = 0$ $\big($Lemma 2.2 (3)$\big)$. Assume now that $l > 0$. Since γ is represented by the torsion class t_0, $\overline{t_l}$ is divisible by 2 in $\mathrm{CH}(\overline{P \times Q})$ (by the compatibility of Steenrod operations with scalar extension). If $l < m - j + i - n_Q$, then $H^{m-j+i-l}$ is also divisible by 2 in $\mathrm{CH}(\overline{Q})$, and so $(H^0 \times H^{m-j+i-l}) \cdot \overline{t_l} \equiv 0 \pmod 4$. To complete the induction step, we can therefore assume that $l \geq m - j + i - n_Q$. Consider now the element

$$\eta := (\pi_P)_*\big((h^0 \times h^{m-j+i-l}) \cdot \gamma\big) \in \mathrm{Ch}^{i-j-l}(P).$$

By construction, η is the mod-2 reduction the torsion integral class

$$\widetilde{\eta} = (\pi_P)_*\big((H^0 \times H^{m-j+i-l}) \cdot t_0\big).$$

We claim that $\widetilde{\eta} \in \mathrm{CH}^{i-j-l}(N)$. Since $t_0 = B - (H^0 \times L_m)$, we have

$$\begin{aligned} \widetilde{\eta} &= (\pi_P)_*\big((H^0 \times H^{m-j+i-l}) \cdot B)\big) - (\pi_P)_*\big((H^0 \times H^{m-j+i-l}) \cdot (H^0 \times L_m)\big) \\ &= (\pi_P)_*\big((H^0 \times H^{m-j+i-l}) \cdot B)\big) - \delta_{i-l,j}(\pi_P)_*(H^0 \times L_0) \\ &= (\pi_P)_*\big((H^0 \times H^{m-j+i-l}) \cdot B)\big) - \delta_{i-l,j} H^0 \end{aligned}$$

Now $H^0 \in \mathrm{CH}^0(N)$ (by the definition of N). At the same time, since we chose B so that $B \circ \nu = B$, we have $(\pi_P)_*\big((H^0 \times H^{m-j+i-l}) \cdot B)\big) \in \mathrm{CH}^{i-j-l}(P) \circ \nu = \mathrm{CH}^{i-j-l}(N)$ by Lemma 10.1 in the appendix. The claim therefore follows from the preceding equality. Since $i - j - l \leq n_Q - m$, it now follows from (iii) that $\deg_{d_P+j-i}(\mu) \equiv 0 \pmod 4$ for any integral representative $\mu \in \mathrm{CH}^{i-j}(P)$ of $S^l(\eta)$. On the other hand, another application of [2, Prop. 61.10], shows that

$$S^l(\eta) = \sum_{r=0}^{l} (\pi_P)_*\Big(\big(h^0 \times c_r(-T_Q)\big) \cdot S^{l-r}\big((h^0 \times h^{m-j+i-l}) \cdot \gamma\big)\Big) \tag{7.3}$$

Now, for each $r \geq 0$, the external product formula for Steenrod operations ([2, Thm. 61.14]) together with Lemma 2.1 gives that

$$S^{l-r}\big((h^0 \times h^{m-j+i-l}) \cdot \gamma\big) = (h^0 \times h^{m-j+i-l}) \cdot S^{l-r}(\gamma) + \left(\sum_{u=1}^{l-r} a_u (h^0 \times h^{m-j+i-l+u}) \cdot S^{l-r-u}(\gamma)\right)$$

for some $a_u \in \mathbb{F}_2$. Since we also have $c_r(-T_Q) = b_r h^r$ for some $b_r \in \mathbb{F}_2$ with $b_0 = 1$ ([2, Lem. 78.1]), it follows from (7.3) that

$$S^l(\eta) = (\pi_P)_*\big((h^0 \times h^{m-j+i-l}) \cdot S^l(\gamma)\big) + \sum_{u=1}^{l} c_u (\pi_P)_*\big((h^0 \times h^{m-j+i-l+u}) \cdot S^{l-u}(\gamma)\big)$$

for some elements $c_u \in \mathbb{F}_2$. Letting d_u be an integral lift of c_u, we get that $S^l(\eta)$ is represented by the integral class

$$(\pi_P)_*\big((H^0 \times H^{m-j+i-l}) \cdot t_l\big) + \sum_{u=1}^{l} d_u (\pi_P)_*\big((H^0 \times H^{m-j+i-l+u}) \cdot t_{l-u}\big) \in \mathrm{CH}_{d_P+j-i}(P)$$

Thus, by the discussion preceding (7.3), the left side of (7.2) (for the given l) is congruent to

$$-\sum_{u=1}^{l} d_u \deg_{d_P+j-i}\Big((\pi_P)_*\big((H^0 \times H^{m-j+i-l+u}) \cdot t_{l-u}\big)\Big)$$

modulo 4. The validity of (7.2) (for the given l) then follows from the induction hypothesis. This completes the proof of the proposition. $\square$

Remark 7.3. As indicated above, the proof of this proposition stems from [24] (and [14, Proof of Thm. 1.2]). The existence of the extension K/F is used here to handle Steenrod operations of smaller degree than those considered (implicitly) in [24]. Note that if X is a smooth projective quadric over a field, and $\eta \in \mathrm{Ch}(X)$ the mod-2 reduction of a torsion element of $\mathrm{CH}(X)$, then a cycle obtained from η by the application of a Steenrod operation need not be represented by an element of degree divisible by 4 (see, e.g., [5, Rem. 5.1]). It is for this reason that we are imposing condition (3) on K/F.

8. Proof of the main theorem

We now proceed to prove our main result. We continue with the set-up and notation of sections 6 and 7. Recall here that $k = \dim((q_{F(p)})_{\mathrm{an}}) = \dim(q) - 2\mathfrak{i}_W(q_{F(p)})$. Since the statement of Conjecture 1.1 holds trivially if $k \geq 2^s - 1$, we will assume for the remainder of this section that $k \leq 2^s - 2$. We can also assume that $\dim(q) > k$ (i.e., that $q_{F(p)}$ is isotropic). As discussed in the proof of Proposition 6.1, it then follows from the separation theorem and Karpenko's theorem on the values of the first higher Witt index ([12], [18, Prop. 10.4]) that $\dim(q) \geq 2^{s+1} - k$. In particular, we have $d_Q > k$. We begin by noting the following reformulation of our conjecture:

Lemma 8.1. *Conjecture 1.1 holds for the pair (p, q) if and only if*

$$\binom{(\dim(q)+k)/2}{j} \equiv 0 \pmod 2 \tag{8.1}$$

for all $\binom{\frac{d_Q+k}{2}}{} - 2^s + 2 \leq j \leq \frac{d_Q-k}{2}$. In this case, the integer a appearing in the statement of the conjecture is even if and only if (8.1) *also holds for $j = \left(\frac{d_Q+k}{2}\right) - 2^s + 1$.*

Proof. If $l = \left(\frac{\dim(q)+k}{2}\right) - j$, then

$$\left(\frac{d_Q+k}{2}\right) - 2^s + 2 \leq j \leq \frac{d_Q-k}{2} \quad \Leftrightarrow \quad k < l < 2^s.$$

The condition in the statement of the lemma is therefore equivalent to the condition that

$$\binom{(\dim(q)+k)/2}{l} \equiv 0 \pmod 2$$

for all $k < l < 2^s$. By a basic fact on the parity of binomial coefficients (see, e.g., [22, Lem. 3.4.2]), however, the latter holds if and only if

$$\frac{\dim(q)+k}{2} = a2^s + \mu$$

for some non-negative integer a and some $0 \leq \mu \leq k$, i.e., if and only if

$$\dim(q) = a2^{s+1} + \epsilon$$

for some non-negative integer a and some $-k \leq \epsilon \leq k$. Furthermore, the integer a is even if and only if $\binom{(\dim(q)+k)/2}{2^s}$ is even, i.e., (8.1) also holds for $j = \left(\frac{d_Q+k}{2}\right) - 2^s + 1$. $\square$

We are going to use Propositions 6.1 and 7.1 to show that the condition of the lemma (and hence the statement of Conjecture 1.1) is satisfied under the hypotheses of Theorem 1.2. To this end, let $\mathcal{B}$ be the standard basis of $\mathrm{Ch}(\overline{P \times Q})$ discussed in §3.A above. Since $\mathfrak{i}_W(q_{F(p)}) = (\dim(q) - k)/2 = (d_Q - k)/2 + 1$, the element $l_{(d_Q-k)/2} \in \mathrm{Ch}(\overline{Q})$ is $F(p)$-rational (Lemma 2.3). By Lemma 3.1, it follows that there exists an F-rational class $\alpha \in \mathrm{Ch}(\overline{P \times Q})$ involving the element $h^0 \times l_{(d_Q-k)/2} \in \mathcal{B}$. Among all such α, let us now fix one involving the *least* number of elements of $\mathcal{B}$ possible. Note then that α is homogeneous of codimension $(d_Q + k)/2$. The basic observation here is the following:

Proposition 8.2. *Suppose, in the above situation, that* $\dim(p) > 2k - 2^{s-1}$*. Then either*

- $\alpha = h^0 \times l_{(d_Q-k)/2}$*, or*
- α *involves the element* $l_{\mathfrak{i}_1(p)-1} \times h^{(d_Q+k)/2-d_P+\mathfrak{i}_1(p)-1} \in \mathcal{B}$*.*

Remarks 8.3. The assumption that $\dim(p) > 2k - 2^{s-1}$ is important for the argument that follows; this (partly) explains the condition on $\dim(p)$ in Theorem 1.2. We also remark here that the second of the two outcomes for α can occur only when p is a maximal splitting form, i.e., when $\mathfrak{i}_1(p) = \dim(p) - 2^s$.

Proof. Assume that α does *not* involve the element $l_{\mathfrak{i}_1(p)-1} \times h^{(d_Q+k)/2-d_P+\mathfrak{i}_1(p)-1}$. It then follows that α involves no element of the form $l_i \times h^j$ with $i < \mathfrak{i}_1(p)$. Indeed, if $\nu \in \mathrm{Ch}_{d_P}(P \times P)$ is (mod-2 reduction of) the idempotent correspondence that determines the upper motive of P, then the minimality property of α gives that $\alpha = \alpha \circ \overline{\nu}$ (note that $\overline{\nu}$ involves $h^0 \times l_0$). Since $\overline{\nu}$ does not involve any terms of the form $l_i \times h^i$ with $i < \mathfrak{i}_1(p) - 1$ (see [23, Thm. 4.13] or [2, §73]), the claim follows.

Now, let $\mathcal{C}$ be the subset of $\mathcal{B}$ consisting of standard basis elements of codimension $(d_Q + k)/2$. Explicitly, $\mathcal{C}$ consists of the following elements:

- $h^i \times h^j$, where $0 \leq i \leq n_P$, $0 \leq j \leq n_Q$ and $i + j = (d_Q + k)/2$;
- $h^i \times l_j$, where $0 \leq i \leq n_Q - (d_Q - k)/2$ and $j - i = (d_Q - k)/2$;
- $l_i \times h^j$, where $0 \leq i \leq n_P$ and $i - j = d_P - (d_Q + k)/2$.

Let $\beta \in \mathcal{C}$ with $\beta \neq h^0 \times l_{(d_Q-k)/2}$. To prove the proposition, we have to show that α does not involve β. If β is F-rational (i.e., if $\beta = h^i \times h^j$ for some i, j), then this is immediate from the minimality property of α. We may therefore assume that β belongs to either the second of third class of elements of $\mathcal{C}$ listed above. Here, we will argue as follows: Assuming that α *does* involve β, we will use the results of §4 and §5 to split off a direct summand in the motive of P (here with $\mathbb{F}_2$ coefficients) having the property that the restriction of α to the complementary direct summand still involves $h^0 \times l_{(d_Q-k)/2}$, but fewer of the remaining elements of $\mathcal{C}$ than α itself (thereby contradicting the minimality property of α). For the sake of simplicity, we will avoid making the motivic interpretation explicit and work only on the level of cycles. In order to make this argument precise, it will be necessary to separate various cases and subcases.

Case 1. $\beta = l_i \times h^j$ for some i, j with $j \leq (d_Q - k)/2$ (and $i - j = d_P - (d_Q + k)/2$). In this case, the element

$$\gamma := \alpha \cdot (h^0 \times h^{(d_Q-k)/2-j}) \in \mathrm{Ch}(\overline{P \times Q})$$

involves $l_i \times h^{(d_Q-k)/2}$, and so the F-rational correspondence $\eta := \gamma^t \circ \alpha \in \mathrm{Ch}_{d_P+i}(\overline{P \times P})$ involves $h^0 \times l_i$. By Proposition 3.4 (1) however, this implies that $i < \mathfrak{i}_1(p)$, and hence contradicts our earlier observation that α involves no terms of the form $l_i \times h^j$ with $i < \mathfrak{i}_1(p)$. We can therefore conclude that α does not involve β in this case.

Case 2. $\beta = h^i \times l_j$ for some i, j satisfying $1 \leq i \leq n_Q - (d_Q - k)/2$ and $j - i = (d_Q - k)/2$. Suppose, to the contrary, that α involves β. Let $F = F_0 \subset F_1 \subset \cdots \subset F_{h(p)}$ be the Knebusch splitting tower of p. By Lemma 3.3 (with $X_1 = Q$ and $X = P$), there then exists $2 \leq l \leq h(p)$ such that $\dim(p_{l-1}) \geq \dim(p) - 2i$ and $\dim((q_{F_l})_{\mathrm{an}}) \leq k - 2i$. In what follows, we fix the smallest such l. We also set $\psi = (q_{F(p)})_{\mathrm{an}}$. Then $\dim(\psi) = k$, and we have $\mathfrak{i}_W(\psi_{F_l}) \geq i$ by the definition of l. We now consider three subcases:

Subcase 2a. $k - 2i \geq 2^{s-1}$. Let $\psi' \subset \psi$ be a subform of dimension $k - i + 1$. since $\mathfrak{i}_W(\psi_{F_l}) \geq i$, ψ' becomes isotropic over F_l. Let $2 \leq r \leq l$ be minimal so that ψ' becomes isotropic over F_r. Since $\dim(\psi') \leq \dim(\psi) = k < 2^s$, the separation theorem then implies that $\dim(p_{r-1}) \leq 2^s$. Let us write $\dim(p_{r-1}) = 2^{s-1} + m$ for some integer $m \leq 2^{s-1}$. We claim that, with r and m as given, the form p satisfies the conditions of Corollary 5.3. First, we have

$$\begin{aligned} m = \dim(p_{r-1}) - 2^{s-1} &\geq \dim(p_{l-1}) - 2^{s-1} \\ &\geq \dim(p) - 2i - 2^{s-1} \\ &> \max(2^s, 2k - 2^{s-1}) - 2i - 2^{s-1} \\ &\geq \max(2^s, 2k - 2^{s-1}) - (k - 2^{s-1}) - 2^{s-1} \\ &= \max(2^s, 2k - 2^{s-1}) - k \\ &\geq 2^{s-2}. \end{aligned}$$

Rearranging the preceding inequalities then gives

$$\begin{aligned} \dim(p) \leq 2i + 2^{s-1} + m \leq k + m& \\ &< 2^s + m \\ &\leq 2^{s-1} + 3m \\ &= \max(2^{s-1} + 3m, 2^s + m). \end{aligned}$$

Now, since ψ' is anisotropic over F_{r-1}, but isotropic over F_r, a theorem of Karpenko and Merkurjev ([16, Thm. 4.1] or [2, Thm. 76.5]) implies that

$$\mathfrak{i}_r(p) = \mathfrak{i}_1(p_{r-1}) \geq \dim(p_{r-1}) - \dim(\psi') + 1 = \dim(p_{r-1}) - k + i.$$

In particular, we have

$$\begin{aligned} \dim(p_r) = \dim(p_{r-1}) - 2\mathfrak{i}_r(p) &\leq \dim(p_{r-1}) - 2(\dim(p_{r-1}) - k + i) \\ &= 2k - (\dim(p_{r-1}) + 2i) \\ &\leq 2k - (\dim(p_{l-1}) + 2i) \\ &\leq 2k - \dim(p) \\ &< 2k - (2k - 2^{s-1}) \\ &= 2^{s-1}. \end{aligned}$$

This proves our claim, so we can conclude that $\mathfrak{i}_r(p) = m$ and that

$$(h^{\mathfrak{j}_{r-1}(p)} \times l_{\mathfrak{j}_r(p)-1}) + (l_{\mathfrak{j}_r(p)-1} \times h^{\mathfrak{j}_{r-1}(p)}) \in \mathrm{Ch}(\overline{P \times P})$$

is F-rational. Now, since

$$\dim(p) - 2i > 2k - 2^{s-1} - 2i > k > 2^{s-1},$$

we have $\dim(p_r) < \dim(p) - 2i \leq \dim(p_{r-1})$, and so $\mathfrak{j}_{r-1}(p) \leq i < \mathfrak{j}_r(p)$. Thus, multiplying the preceding cycle class through by $h^i \times h^{\mathfrak{j}_r(p)-i-1} \in \mathrm{Ch}(\overline{P \times P})$, we get that the correspondence

$$\eta = (h^i \times l_i) + (l_{\mathfrak{j}_r(p)-i-1} \times h^{\mathfrak{j}_r(p)-i-1}) \in \mathrm{Ch}_{d_P}(\overline{P \times P})$$

is F-rational. In particular, the element

$$\alpha - \alpha \circ \eta \in \mathrm{Ch}(\overline{P \times Q})$$

is F-rational. Note, however, that this contradicts the minimality property of α. Indeed, since $i > 0$, the constructed element involves $h^0 \times l_{(d_Q-k)/2}$, but not $\beta = h^i \times l_j$ nor any element of $\mathcal{B}$ that is not involved in α. We conclude that this subcase cannot occur.

Subcase 2b. $2^s - k \leq k - 2i < 2^{s-1}$. We claim in this subcase that there exists a field extension K/F such that if $\sigma = (q_K)_{\mathrm{an}}$ and $\tau = (p_K)_{\mathrm{an}}$, then:

(a) $2^{s-1} < \dim(\sigma), \dim(\tau) \leq 2^s$;
(b) $\dim(\sigma) - \mathfrak{i}_1(\sigma) = 2^{s-1}$; and
(c) $\dim((\sigma_{K(\tau)})_{\mathrm{an}}) \leq k - 2i < 2^{s-1}$.

Note first that $2^{s-1} + i \leq k < 2^s$ by assumption. Since $i > 0$, and since $\dim(\psi) = k$, the splitting pattern of q therefore contains an integer in the open interval $(2^{s-1}, 2^s)$. We can now define K as follows: Take K' to be the largest field in the Knebusch splitting tower of q for which $(q_{K'})_{\mathrm{an}}$ has dimension $> 2^{s-1}$, and then take K to be the largest field in the Knebusch splitting tower of $p_{K'}$ over which $(q_{K'})_{\mathrm{an}}$ remains anisotropic. Define σ and τ as above. It is then clear that $2^{s-1} < \dim(\sigma) < 2^s$, and since $\sigma_{K(\tau)}$ is isotropic (by the choice of K), the separation theorem then gives that $\dim(\tau) \leq 2^s$. On the other hand, since the dimension of the anisotropic part of q over the composite field $K \cdot F_l$ is at most $k - 2i < 2^{s-1}$, σ becomes isotropic over $K \cdot L_r$. By the definition of K, it follows that $\dim(\tau) \geq \dim(p_{l-1})$. But

$$\dim(p_{l-1}) \geq \dim(p) - 2i > 2k - 2^{s-1} - 2i \geq 2(2^{s-1} + i) - 2^{s-1} - 2i = 2^{s-1},$$

and so condition (a) is satisfied, i.e., $2^{s-1} < \dim(\sigma), \dim(\tau) \leq 2^s$. At the same time, if L/K is a field extension for which $\dim((q_L)_{\mathrm{an}}) \leq 2^{s-1}$, then we must have that $\dim((q_L)_{\mathrm{an}}) \leq k - 2i$. Indeed, if $k - 2i < \dim((q_L)_{\mathrm{an}}) \leq 2^{s-1}$, then $\dim((q_L)_{\mathrm{an}})$ would become isotropic over the composite field $L \cdot F_l$, thereby contradicting the separation theorem (since $\dim(p_{l-1}) > 2^{s-1}$). It follows that (c) is satisfied. Moreover, since $\dim(\sigma_1) \leq 2^{s-1}$ by the definition of K, we get that $\dim(\sigma_1) \leq k - 2i < 2^{s-1}$. By Karpenko's theorem on the values of the first higher Witt index ([12], [18, Prop. 10.4]), it follows that $\dim(\sigma) - \mathfrak{i}_1(\sigma) = 2^{s-1}$, i.e., condition (b) is also satisfied, and so our claim is proved. By Theorem 4.3, it follows that the forms σ and τ are stably birational equivalent, and that $\dim(\tau) - \mathfrak{i}_1(\tau) = \dim(\sigma) - \mathfrak{i}_1(\sigma) = 2^{s-1}$ (see Proposition 4.1). Now, since $\dim(\tau) = \dim(p) - 2\mathfrak{i}_W(p_K)$, the inequalities

$$\dim(p) - 2i \leq \dim(p_{l-1}) \leq \dim(\tau) \qquad \text{and} \qquad 2^s - \dim(\tau) < 2^{s-1} < \dim(p) - 2i$$

imply that

$$\mathfrak{i}_W(p_K) \leq i < \dim(p) - 2^{s-1} - \mathfrak{i}_W(p_K).$$

Similarly, since $\dim(\sigma) = \dim(q) - 2\mathfrak{i}_W(q_K)$, and since $j = i + (d_Q - k)/2$, the inequalities

$$k - 2i < 2^{s-1} < \dim(\sigma) \qquad \text{and} \qquad 2^s - \dim(\sigma) = \dim(\sigma_1) \leq k - 2i$$

give that

$$\mathfrak{i}_W(q_K) \leq j < \dim(q) - 2^{s-1} - \mathfrak{i}_W(q_K).$$

By Proposition 4.2 (with $d = 2^{s-1}$), it follows that α involves the standard basis element

$$l_{\dim(p) - 2^{s-1} - i - 1} \times h^{\dim(q) - 2^{s-1} - j - 1}.$$

Noting that

$$i \leq \dim(p) - 2^{s-1} - i - 1 \qquad \text{and} \qquad \dim(q) - 2^{s-1} - j - 1 \leq j.$$

(the first inequality says that $\dim(p) - 2i > 2^{s-1}$, and the second that $k - 2i < 2^{s-1}$), we can then consider the F-rational element

$$\gamma = \alpha \cdot (h^{(\dim(p) - 2^{s-1} - i - 1) - i} \times h^{j - (\dim(q) - 2^{s-1} - j - 1)}) \in \mathrm{Ch}(\overline{P \times Q}).$$

Now, by construction, the correspondence $\eta = \gamma^t \circ \alpha \in \mathrm{Ch}_{d_P}(\overline{P \times P})$ involves the standard basis element $h^i \times l_i$, but does not involve $h^0 \times l_0$. Indeed, in order for η to involve $h^0 \times l_0$, we would require α to involve a term of the form $l_a \times h^b$ with $b \leq (d_Q - k)/2$ (see the definition of γ). By Case 2, however, α does not involve any such terms. As a result, we see that the F-rational cycle class

$$\alpha - \alpha \circ \eta \in \mathrm{Ch}(\overline{P \times Q})$$

involves $h^0 \times l_{(d_Q-k)/2}$, but not $\beta = h^i \times l_j$ nor any element of $\mathcal{B}$ that is not involved in α. As this contradicts the minimality property of α, we can also conclude that this subcase does not occur.

Subcase 2c. $k - 2i < \min(2^s - k, 2^{s-1})$. In this subcase, we claim that there exists a field extension K/F such that if $\sigma = (q_K)_{\mathrm{an}}$ and $\tau = (p_K)_{\mathrm{an}}$, then:

(a) $2^{s-2} < \dim(\sigma), \dim(\tau) \leq 2^{s-1}$;
(b) $\dim(\sigma) - \mathfrak{i}_1(\sigma) = 2^{s-2}$; and
(c) $\dim((\sigma_{K(\tau)})_{\mathrm{an}}) \leq k - 2i < 2^{s-2}$.

We first observe that the splitting pattern of q contains an integer in the interval $(k - 2i, 2^{s-1})$. This is clear if $k = \dim(\psi) < 2^s$, so we can assume that $2^{s-1} \leq \dim(\psi) < 2^s$. Let $0 \leq m \leq k - 2^{s-1}$ be such that $2^{s-1} + m$ is the smallest integer $\geq 2^{s-1}$ in the splitting pattern of q. The theorem on the values of the first higher Witt index then implies that the next smallest integer in the splitting pattern of q is $2^{s-1} - m$, and this must be $\leq k - 2i$ by assumption. But this implies that

$$k - 2i \geq 2^{s-1} - m \geq 2^{s-1} - (k - 2^{s-1}) = 2^s - k,$$

contrary to the standing hypothesis. Thus, there exists an extension F'/F such that $\dim((q_{F'})_{\mathrm{an}}) \in (k - 2i, 2^{s-1})$. Since $(q_{F'\cdot F_l})_{\mathrm{an}}$ has dimension at most $k - 2i$, the separation theorem then implies that $\dim(p_{l-1}) \leq 2^{s-1}$. In particular, we have that $\dim(p) - 2i \leq 2^{s-1}$. Since $\dim(p) > 2^s$, it follows that $2i > 2^{s-1}$, and so $k - 2i < \min(2^s - k, k - 2^{s-1}) \leq 2^{s-2}$. Now, we have

$$\begin{aligned}
\dim(p) - 2i &> \max(2k - 2^{s-1}, 2^s) - 2i \\
&\geq \max(2k - 2^{s-1}, 2^s) - 2\big(n_Q - (d_Q - k)/2\big) \\
&\geq \max(2k - 2^{s-1}, 2^s) - k \\
&= \max(k - 2^{s-1}, 2^s - k) \\
&\geq 2^{s-2},
\end{aligned}$$

so that $2^{s-2} < \dim(p) - 2i \leq \dim(p_{l-1}) \leq 2^{s-1}$. By another application of the separation theorem, it follows that the splitting pattern of q does not contain an integer in the interval $(k - 2i, 2^{s-2}]$, and hence contains an integer in the open interval $(2^{s-2}, 2^{s-1})$. In summary, we have established the following:

- $k - 2i < 2^{s-2}$.
- $2^{s-2} < \dim(p) - 2i \leq \dim(p_{l-1}) \leq 2^{s-1}$.
- The splitting pattern q contains an integer in the open interval $(2^{s-2}, 2^{s-1})$.

Using these observations, the same arguments used in subcase 2b now show the existence of an extension K/F for which $\sigma = (q_K)_{\mathrm{an}}$ and $\tau = (p_K)_{\mathrm{an}}$ satisfy conditions (a)-(c) above; explicitly, we let K' be the largest field in the Knebusch splitting tower of q for which $\dim((q_{K'})_{\mathrm{an}}) > 2^{s-2}$, and then take K to be the largest field in the Knebusch splitting tower of $p_{K'}$ over which $(q_{K'})_{\mathrm{an}}$ remains anisotropic. With this choice of K, we again have that $\dim(\tau) \geq \dim(p_{l-1})$. By Theorem 4.3, it follows that σ and τ are stably birational equivalent, and that $\dim(\tau) - \mathfrak{i}_1(\tau) = \dim(\sigma) - \mathfrak{i}_1(\sigma) = 2^{s-2}$ (see Proposition 4.1). Now, since $\dim(\tau) = \dim(p) - 2\mathfrak{i}_W(p_K)$, the inequalities

$$\dim(p) - 2i \leq \dim(p_{l-1}) \leq \dim(\tau) \qquad \text{and} \qquad 2^{s-1} - \dim(\tau) < 2^{s-2} < \dim(p) - 2i$$

imply that

$$\mathfrak{i}_W(p_K) \leq i < \dim(p) - 2^{s-2} - \mathfrak{i}_W(p_K).$$

Similarly, since $\dim(\sigma) = \dim(q) - 2\mathfrak{i}_W(q_K)$, and since $j = i + (d_Q - k)/2$, the inequalities

$$k - 2i < 2^{s-2} < \dim(\sigma) \qquad \text{and} \qquad 2^{s-1} - \dim(\sigma) = \dim(\sigma_1) \leq k - 2i$$

(for the second inequality, recall that the splitting pattern of q contains no integer in the interval $(k - 2i, 2^{s-2}]$) imply that

$$\mathfrak{i}_W(q_K) \leq j < \dim(q) - 2^{s-1} - \mathfrak{i}_W(q_K).$$

Proposition 4.2 (with $d = 2^{s-2}$) therefore implies that α involves the standard basis element

$$l_{\dim(p)-2^{s-2}-i-1} \times h^{\dim(q)-2^{s-2}-j-1}.$$

Noting that

$$i \leq \dim(p) - 2^{s-2} - i - 1 \qquad \text{and} \qquad \dim(q) - 2^{s-2} - j - 1 \leq j$$

(the first inequality says that $\dim(p) - 2i > 2^{s-2}$ and the second that $k - 2i < 2^{s-2}$), we can then consider the F-rational element

$$\gamma = \alpha \cdot (h^{(\dim(p)-2^{s-1}-i-1)-i} \times h^{j-(\dim(q)-2^{s-1}-j-1)}) \in \mathrm{Ch}(\overline{P \times Q}).$$

As in subcase 2b, the composition $\eta = \gamma^t \circ \alpha \in \mathrm{Ch}_{d_P}(\overline{P \times P})$ then involves the standard basis element $h^i \times l_i$, but does not involve $h^0 \times l_0$. In particular, the F-rational element

$$\alpha - \alpha \circ \eta \in \mathrm{Ch}(\overline{P \times Q})$$

involves $h^0 \times l_{(d_Q-k)/2}$, but not $\beta = h^i \times l_j$ nor any element of $\mathcal{B}$ that is not involved in α. Since this contradicts the minimality property of α, we again conclude that this subcase cannot occur.

Since subcases 2a-2c cover all possible values of $k - 2i$, we conclude that α does not involve $\beta = h^i \times l_j$ (i.e., case 2 does not occur). We are thus left with:

Case 3. $\beta = l_i \times h^j$ for some i, j satisfying $1 \leq i \leq n_P$, $i - j = d_P - (d_Q + k)/2$ and $j > (d_Q - k)/2$ (recall that the case where $j \leq (d_Q - k)/2$ was dealt with in Case 2 above). Suppose to the contrary that α involves β, and let $F = F_0 \subset F_1 \subset \cdots \subset F_{h(q)}$ now denote the Knebusch splitting pattern of q (as opposed to p). By Lemma 3.3 (now with $X_1 = P$ and $X = Q$), there then exists $1 \leq l \leq h(q)$ such that $\dim((p_{F_l})_{\mathrm{an}}) \leq d_P - 2i$ and $\dim(q_{l-1}) \geq \dim(q) - 2j = (d_P - 2i) + (\dim(p) - k)$. In what follows, we fix the smallest such l. Note that the inequality $j > (d_Q - k)/2$ is equivalent to $\dim(q) - 2j \leq k$. On the other hand, we have

$$\dim(q) - 2j \geq \dim(p) - k > \max(k - 2^{s-1}, 2^s - k) \geq 2^{s-2},$$

and so $2^{s-2} < \dim(q) - 2j \leq k < 2^s$. Lastly, note that we then have

$$d_P - 2i = (\dim(q) - 2j) - (\dim(p) - k) \leq 2k - \dim(p) < 2^{s-1}.$$

We will now consider two subcases:

Subcase 3a. $\dim(q) - 2j > 2^{s-1}$ (in particular, $\dim(q_{l-1}) > 2^{s-1}$). Let $1 \leq r \leq h(p)$ be minimal so that $\dim(p_r) \leq d_P - 2i$. Since $\dim((q_{F_l})_{\mathrm{an}}) \leq d_P - 2i$, and since $\dim(q_{l-1}) > 2^{s-1}$ by assumption, the separation theorem implies that $\dim(p_{r-1}) > 2^{s-1}$. Suppose that $\dim(p_{r-1}) > 2^s$. Since $\dim(p) \leq 2^{s+1}$, we can then write $\dim(p_{r-1}) = 2^s + m$ for some $1 \leq m \leq 2^s$. Now, since $\dim(p_r) \leq d_P - 2i < 2^{s-1}$, the theorem on the values of the first higher Witt index implies that $\mathfrak{i}_r(p) = \mathfrak{i}_1(p_{r-1}) = m > 2^{s-1}$. In particular, we have $\dim(p) \leq 2^{s+1} < \min(2^s + 3m, 2^{s+1} + m)$ and $\mathfrak{i}_t(p) \leq \dim(p_r)/2 < 2^{s-2} < \mathfrak{i}_r(p)$ for all $r < t \leq h(p)$. By Corollary 5.3, it follows that the correspondence

$$(h^{\mathfrak{j}_{r-1}(p)} \times l_{\mathfrak{j}_r(p)-1}) + (l_{\mathfrak{j}_r(p)-1} \times h^{\mathfrak{j}_{r-1}(p)}) \in \mathrm{Ch}(\overline{P \times P})$$

is F-rational (this is equivalent to saying that the rth shell of P consists of binary motives; see Proposition 5.5 and the remarks that precede it). Now, since $\dim(p_r) \leq d_P - 2i < \dim(p_{r-1})$, we have $\mathfrak{j}_{r-1}(p) < i < \mathfrak{j}_r(p)$, and so multiplying the above correspondence by $h^{\mathfrak{j}_r(p)-i-1} \times h^{i-\mathfrak{j}_{r-1}(p)} \in \mathrm{Ch}(\overline{P \times P})$, we get that

$$\eta = (h^{\mathfrak{j}_{r-1}(p)+\mathfrak{j}_r(p)-i-1} \times l_{\mathfrak{j}_{r-1}(p)+\mathfrak{j}_r(p)-i-1}) + (l_i \times h^i) \in \mathrm{Ch}(\overline{P \times P})$$

is F-rational. If $\mathfrak{j}_{r-1}(p) + \mathfrak{j}_r(p) - i - 1 = 0$ $\big($i.e., $\pi = (h^0 \times l_0) + (l_i \times h^i)\big)$, then we must have that $r = 1$, and Proposition 4.1 (2) then implies that $i = \mathfrak{i}_1(p) - 1 = \dim(p) - 2^s - 1$. But since $d_P - 2i \leq 2k - \dim(p)$, we have

$$i \geq \dim(p) - (k+1) > \dim(p) - 2^s,$$

and so this is not the case. On the other hand, if $\mathfrak{j}_{r-1}(p) + \mathfrak{j}_r(p) - i - 1 > 0$, then the F-rational element

$$\alpha - \alpha \circ \eta \in \mathrm{Ch}(\overline{P \times Q})$$

involves $h^0 \times l_{(d_Q-k)/2}$, but not $\beta = l_i \times h^i$ nor any element of $\mathcal{B}$ that is not involved in α. Since this contradicts the minimality property of α, we conclude that $\dim(p_{r-1}) \not> 2^s$, i.e., $2^{s-1} < \dim(p_{r-1}) \leq 2^s$. With this established, let K' be the field of definition of p_{r-1}, let K be the largest entry of the Knebusch splitting tower of $q_{K'}$ over which p_{r-1} remains anisotropic, and let $\sigma = (q_K)_{\mathrm{an}}$ and $\tau = (p_K)_{\mathrm{an}}$. We claim that:

(a) $2^{s-1} < \dim(\sigma), \dim(\tau) \leq 2^s$;
(b) $\dim(\tau) - \mathfrak{i}_1(\tau) = 2^{s-1}$; and
(c) $\dim((\tau_{K(\sigma)})_{\mathrm{an}}) \leq d_P - 2i < 2^{s-1}$.

Since $\dim(\tau) = \dim(p_{r-1})$ by definition, we have $2^{s-1} < \dim(\tau) \leq 2^{s+1}$. Moreover, since

$$\dim(\tau_1) \leq \dim(p_r) \leq d_P - 2i,$$

the specialization results of Knebusch ([17, §3]) imply that $\dim((\tau_L)_{\mathrm{an}}) \leq d_P - 2i < 2^{s-1}$ for any field extension L/K with the property that τ_L is isotropic. In particular, this applies to $L = K(\sigma)$ ($\tau_{K(\sigma)}$ is isotropic by the definition of K), and so (c) holds. At the same time, it also applies to $L = K(\tau)$, and statement (b) is then a consequence of the theorem on the values of the first higher Witt index. With (a)-(c) established, it now follows from Theorem 4.3 that σ and τ are stably birational equivalent, and that $\dim(\sigma) - \mathfrak{i}_1(\sigma) = \dim(\tau) - \mathfrak{i}_1(\tau) = 2^{s-1}$ (see Proposition 4.1). Now, since $\dim(\tau) = \dim(p) - 2\mathfrak{i}_W(p_K)$, the inequalities

$$\dim(\tau) = \dim(p_{r-1}) > d_P - 2i \qquad \text{and} \qquad 2^s - \dim(\tau) = \dim(\tau_1) \leq d_P - 2i$$

imply that

$$\mathfrak{i}_W(p_K) \leq i < \dim(p) - 2^{s-1} - \mathfrak{i}_W(p_K).$$

At the same time, we also have

$$\mathfrak{i}_W(q_K) \leq j < \dim(q) - 2^{s-1} - \mathfrak{i}_W(q_K).$$

Indeed, since $\dim((p_{F_l})_{\mathrm{an}}) \leq d_P - 2i$, we have $\dim(\sigma) \geq \dim(q_{l-1}) > \dim(q) - 2j$ by the definition of K. Since $\dim(\sigma) = \dim(q) - 2\mathfrak{i}_W(q_K)$, this gives the first inequality, and the second follows from the fact that $2^s - \dim(\sigma) < 2^{s-1} \leq \dim(q) - 2j$. In view of Proposition 4.2, we conclude from the preceding discussion that α involves the standard basis element

$$h^{\dim(p)-2^{s-1}-i-1} \times l_{\dim(q)-2^{s-1}-j-1}.$$

By Case 2, this is possible only if $\dim(p) - 2^{s-1} - i - 1 = 0$. But the latter equality implies that

$$2i = 2\dim(p) - 2^s - 2 > \dim(p) - 2 = d_P,$$

which is impossible. We can therefore conclude that this subcase does not occur.

Subcase 3b. $\dim(q) - 2j \leq 2^{s-1}$. In this subcase, we have that

$$d_P - 2i = (\dim(q) - 2j) - (\dim(p) - k) \leq 2^{s-1} - \max(k - 2^{s-1}, 2^s - k) < 2^{s-2}.$$

As in the previous subcase, let $1 \leq r \leq h(p)$ be minimal so that $\dim(p_r) \leq d_P - 2i$. Since $\dim((p_{F_l})_{\mathrm{an}}) \leq d_P - 2i < 2^{s-2}$, and since $\dim(q_{l-1}) > 2^{s-2}$, the separation theorem implies that $\dim(p_{r-1}) > 2^{s-2}$. Suppose that $\dim(p_{r-1}) > 2^{s-1}$, and write $\dim(p_{r-1}) = 2^n + m$ for unique integers $n \in \{s-1, s\}$ and $1 \leq m \leq 2^n$. Since $\dim(p_r) < 2^{s-2}$, the theorem on the values of the first higher Witt index gives that $\mathfrak{i}_r(p) = \mathfrak{i}_1(p_{r-1}) = m > 2^{s-2}$. In particular, $m > 2^{n-1}$ and so $\min(2^n + 3m, 2^{n+1} + m) = 2^{n+1} + m$. If $\dim(p) > 2^{n+1} + m$, then

$$\begin{aligned}
\dim(q) - 2j &= (d_P - 2i) + (\dim(p) - k) \\
&> \dim(p_r) + (2^{n+1} + m - k) \\
&= (2^n - m) + (2^{n+1} + m - k) \\
&= 2^n + (2^{n+1} - k) \\
&\geq 2^n + (2^s - k) \\
&> 2^n,
\end{aligned}$$

contrary to our assumption that $\dim(q) - 2j \leq 2^{s-1}$. We therefore have that $\dim(p) > 2^{n+1} + m = \min(2^n + 3m, 2^{n+1} + m)$. Since $\mathfrak{i}_t(p) \leq \dim(p_r)/2 < 2^{s-3} < m = \mathfrak{i}_r(p)$ for all $r < t \leq h(p)$, Corollary 5.3 now implies that the correspondence

$$(h^{\mathfrak{j}_{r-1}(p)} \times l_{\mathfrak{j}_r(p)-1}) + (l_{\mathfrak{j}_r(p)-1} \times h^{\mathfrak{j}_{r-1}(p)}) \in \mathrm{Ch}(\overline{P \times P})$$

is F-rational (again, this is equivalent to saying that the rth shell of P consists of binary motives). Now, since $\dim(p_r) \leq d_P - 2i < \dim(p_{r-1})$, we have $\mathfrak{j}_{r-1}(p) < i < \mathfrak{j}_r(p)$, and so multiplying the above correspondence by $h^{\mathfrak{j}_r(p)-i-1} \times h^{i-\mathfrak{j}_{r-1}(p)} \in \mathrm{Ch}(\overline{P \times P})$, we get that

$$\eta = (h^{\mathfrak{j}_{r-1}(p)+\mathfrak{j}_r(p)-i-1} \times l_{\mathfrak{j}_{r-1}(p)+\mathfrak{j}_r(p)-i-1}) + (l_i \times h^i) \in \mathrm{Ch}(\overline{P \times P})$$

is F-rational. If $\mathfrak{j}_{r-1}(p) + \mathfrak{j}_r(p) - i - 1 = 0$ $\big($i.e., $\pi = (h^0 \times l_0) + (l_i \times h^i)\big)$, then we must have that $r = 1$, and Proposition 3.4 (2) then implies that $i = \mathfrak{i}_1(p) - 1 = \dim(p) - 2^n - 1$. This is impossible: indeed, as noted in the previous subcase, we have $i > \dim(p) - 2^s$. In particular, the stated equality implies that $\mathfrak{i}_1(p) = \dim(p) - 2^{s-1}$, thereby contradicting the separation theorem (the latter implies that $\mathfrak{i}_1(p) \leq \dim(p) - 2^s$). On the other hand, if $\mathfrak{j}_{r-1}(p) + \mathfrak{j}_r(p) - i - 1 > 0$, then the F-rational element

$$\alpha - \alpha \circ \eta \in \mathrm{Ch}(\overline{P \times Q})$$

involves $h^0 \times l_{(d_Q-k)/2}$, but not $\beta = l_i \times h^j$ nor any element of $\mathcal{B}$ that is not involved in α. Since this contradicts the minimality property of α, we conclude that $\dim(p_{r-1}) \not> 2^{s-1}$, and so $2^{s-2} < \dim(p_{r-1}) \leq 2^{s-1}$. Now, let K' be the field of definition of p_{r-1}, let K be the largest entry in the Knebusch splitting tower of $q_{K'}$ over which p_{r-1} remains anisotropic, and let $\sigma = (q_K)_{\mathrm{an}}$ and $\tau = (p_K)_{\mathrm{an}}$. The same argument used in subcase 3a above then gives that:

(a) $2^{s-2} < \dim(\sigma), \dim(\tau) \leq 2^{s-1}$;
(b) $\dim(\tau) - \mathfrak{i}_1(\tau) = 2^{s-2}$; and
(c) $\dim((\tau_{K(\sigma)})_{\mathrm{an}}) \leq d_P - 2i < 2^{s-2}$.

By Theorem 4.3, it follows that σ and τ are stably birational equivalent, and that $\dim(\sigma) - \mathfrak{i}_1(\sigma) = \dim(\tau) - \mathfrak{i}_1(\tau) = 2^{s-2}$ (see Proposition 4.1). Now, since $\dim(\tau) = \dim(p) - 2\mathfrak{i}_W(p_K)$, the inequalities

$$\dim(\tau) = \dim(p_{r-1}) > d_P - 2i \qquad \text{and} \qquad 2^{s-1} - \dim(\tau) = \dim(\tau_1) \leq d_P - 2i$$

imply that

$$\mathfrak{i}_W(p_K) \leq i < \dim(p) - 2^{s-2} - \mathfrak{i}_W(p_K).$$

At the same time, we also have that

$$\mathfrak{i}_W(q_K) \leq j < \dim(q) - 2^{s-2} - \mathfrak{i}_W(q_K).$$

Indeed, since $\dim((p_{F_l})_{\mathrm{an}}) \leq d_P - 2i$, we have $\dim(\sigma) \geq \dim(q_{l-1}) \geq \dim(q) - 2j$ by the definition of K. Since $\dim(\sigma) = \dim(q) - 2\mathfrak{i}_W(q_K)$, this gives the first inequality, and the second follows from the fact that $2^{s-1} - \dim(\sigma) < 2^{s-2} < \dim(q) - 2j$. In view of Proposition 4.2, we conclude from the preceding discussion that α involves the standard basis element

$$h^{\dim(p) - 2^{s-2} - i - 1} \times l_{\dim(q) - 2^{s-2} - j - 1}.$$

By Case 2, this is possible only if $\dim(p) - 2^{s-2} - i - 1 = 0$. But the latter equality implies that

$$2i = 2\dim(p) - 2^{s-1} - 2 > \dim(p) - 2 = d_P,$$

which is impossible. We can therefore conclude that this subcase does not occur.

Since $\dim(q) - 2j \leq 2^{s-1}$, subcases 3a and 3b cover all possible values of j, and so we can conclude that α does not involve $\beta = l_i \times h^j$, i.e., case 3 does not occur. This completes the proof of the proposition. $\square$

Proposition 8.2 is the first step towards being able to apply the technical Proposition 7.1 to our situation. We still need, however, to be able to deal with condition (3) in the statement of the latter. To this end, we first note that this condition is automatically satisfied if the codimension of the given cycle class is small enough:

Lemma 8.4. *Let Y be a smooth projective quadric over a field L, and let i be an integer $\leq \frac{\dim(Y)}{4}$. If $\gamma \in \mathrm{Ch}^i(Y)$ is represented by a torsion element of $\mathrm{CH}^i(Y)$, then $S^j(\gamma) \in \mathrm{Ch}^{i+j}(Y)$ is represented by a torsion element of $\mathrm{CH}^{i+j}(Y)$ for any $j \geq 0$.*

Proof. By [2, Thm. 61.13], we have

$$S^j(\gamma) = \begin{cases} \gamma^2 & \text{if } j = i \\ 0 & \text{if } j > i, \end{cases}$$

and so the statement holds when $j \geq i$. Assume now that $j < i$. Since γ is represented by a torsion element of $\mathrm{CH}^i(Y)$, and since Steenrod operations commute with scalar extension, we have $\overline{S^j(\gamma)} = 0$ in $\mathrm{Ch}^{i+j}(\overline{Y})$. In particular, if $\mu \in \mathrm{CH}^{i+j}(Y)$ is an integral representative of $S^j(\gamma)$, then $\overline{\mu} \in 2\mathrm{CH}^{i+j}(\overline{Y})$. On the other hand, since $j < i$, the inequality $i \leq \frac{\dim(Y)}{4}$

implies that $i+j<\dim(Y)/2$, and so $\overline{\mu}=2aH^{i+j}$ for some integer a (see §2.C above). The element $\mu-2aH^{i+j}\in \mathrm{CH}^{i+j}(Y)$ is then a torsion class representing $S^j(\gamma)$. □

Remark 8.5. We cannot avoid making some assumption on the codimension of γ here (cf. Remark 7.3).

Now, to handle condition (3) of Proposition 7.1, we will use the following consequence of this lemma:

Lemma 8.6. *Let L, Y, i and γ be as in Lemma 8.4. Let φ be a quadratic form of dimension ≥ 2 over L, and let X denote its associated (smooth) projective quadric. Suppose we are given a mod-2 Chow correspondence $\theta\in \mathrm{Ch}_d(Y\times X)$ (for some $d\geq 0$) and an integral representative σ of the element $S^b\big(\theta_*(\gamma)\big)\in \mathrm{Ch}_{d-i-b}(X)$, where b is a non-negative integer. If $\mathfrak{i}_W(\varphi)\leq d-i-b$, then $\deg_{d-i-b}(\sigma)\equiv 0 \pmod 4$.*

Proof. Since $\mathfrak{i}_W(\varphi)\leq d-i-b$, we have $\deg_{d-i-b}\big(\mathrm{CH}_{d-i-b}(X)\big)=2\mathbb{Z}$ (Lemma 2.3). In view of Lemma 2.2 (3), it thus suffices to show that $S^b\big(\theta_*(\gamma)\big)$ is represented by a torsion element of $\mathrm{CH}_{d-i-b}(X)$. This follows immediately from Lemma 8.4 together with Lemma 10.2 in the appendix below. □

We will now apply this to our given form p over F. Recall again that $U(P)$ denotes the upper motive of the quadric P (see §3.E). The result we need here is the following:

Proposition 8.7. *Suppose, in the above situation, that $\mathrm{char}(F)\neq 2$, and that K/F is a field extension preserving the anisotropy of p and all of its higher Witt indices. Let a be an integer $\leq \min\Big(\Big(\frac{d_P+1}{4}\Big)+2^{s-3}, 2^{s-1}-1\Big)$, and let $\eta\in \mathrm{Ch}^a(P_K)$ be the mod-2 reduction of a torsion element of $\mathrm{CH}^a(U(P)_K)$. Then $\deg_{d_P-a-b}(\mu)\equiv 0 \pmod 4$ for any pair (b,μ) consisting of an integer $b\geq 0$ and an integral representative $\mu\in \mathrm{CH}^{a+b}(P_K)$ of $S^b(\eta)\in \mathrm{Ch}^{a+b}(P_K)$.*

Proof. Let us assume first that $K=F$. As in the proof of Lemma 8.4, [2, Thm. 61.13] gives that $S^b(\eta)$ is represented by a torsion element of $\mathrm{CH}^{a+b}(P)$ for all $b\geq a$. Thus, for all such b, the congruence $\deg_{d_P-a-b}(\mu)\equiv 0 \pmod 4$ follows from Lemma 2.2 (3) and Lemma 2.3 (remember that p is anisotropic). Assume now that $b<a$, so that

$$\begin{aligned}
d_P-a-b &\geq d_P-2a+1\\
&\geq d_P-2\left(\min\left(\left(\frac{d_P+1}{4}\right)+2^{s-3}\right),2^{s-1}-1\right)+1\\
&= \max\left(\left(\frac{d_P-1}{2}\right)-2^{s-2}+1, d_P-2^s+3\right). \qquad (8.2)
\end{aligned}$$

Let $F=F_0\subset F_1\subset\cdots\subset F_{h(p)}$ be the Knebusch splitting tower of p, and let r be the *largest* non-negative integer for which $\mathfrak{j}_{r-1}(p)\leq d_P-a-b$. Let X' be the projective quadric defined by the vanishing of p_{r-1} over F_{r-1}. Considering the motivic decomposition (3.1) for $P_{F_{r-1}}$, we find a correspondence $in\colon \mathrm{CH}_{d_P-\mathfrak{j}_{r-1}(p)}(X'\times P_{F_{r-1}})$ with the property that the push-forward $(in)_*\colon \mathrm{CH}(X')\to \mathrm{CH}(P_{F_{r-1}})$ identifies the torsion subgroups of $\mathrm{CH}(X')$ and $\mathrm{CH}(P_{F_{r-1}})$ (this is the correspondence $in^{P_{F_{r-1}}}$ considered in §2.E). Passing to Chow groups modulo 2, it follows that $\eta_{F_{r-1}}=(in)_*(\gamma_0)$, where $\gamma_0\in \mathrm{Ch}^{a-\mathfrak{j}_{r-1}(p)}(X')$ is the mod-2 reduction of a torsion element of $\mathrm{CH}^{a-\mathfrak{j}_{r-1}(p)}(X')$ (and in now denotes the mod-2 reduction of the integral correspondence considered above). We separate two cases.

Case 1. $\dim(p_{r-1}) \leq 2d_P - 4a + 2$. In this case, we have

$$\begin{aligned} 4\big(a - \mathfrak{j}_{r-1}(p)\big) &= 4\left(a - \left(\frac{\dim(p) - \dim(p_{r-1})}{2}\right)\right) \\ &= 4a - 2\dim(p) + 2\dim(p_{r-1}) \\ &= 4a - 2d_P + 2\dim(p_{r-1}) - 4 \\ &\leq \dim(p_{r-1}) - 2 \\ &= \dim(X'). \end{aligned}$$

At the same time, we have

$$\begin{aligned} \mathfrak{i}_W(p_{F_{r-1}}) = \mathfrak{j}_{r-1}(p) &\leq d_P - a - b \\ &= \big(d_P - \mathfrak{j}_{r-1}(p)\big) - \big(a - \mathfrak{j}_{r-1}(p)\big) - b, \end{aligned}$$

and so the conditions of Lemma 8.6 are satisfied for the following data:

$$L = F_{r-1};\ \ Y = X';\ \ \varphi = p_{F_{r-1}};\ \ X = P_{F_{r-1}};\ \ i = a - \mathfrak{j}_{r-1}(p);$$
$$d = d_P - \mathfrak{j}_{r-1}(p);\ \ \gamma = \gamma_0;\ \ \theta = in;\ \ \sigma = \eta_{F_{r-1}}.$$

It follows that $\deg_{d_P-a-b}(\mu_{F_{r-1}}) \equiv 0 \pmod 4$, and since $\deg_{d_P-a-b}(\mu)$ does not change under scalar extension, this proves what we want.

Case 2. $\dim(p_{r-1}) \geq 2d_P - 4a + 3$. In this case, we have

$$\begin{aligned} \dim(p_{r-1}) &\geq 2d_P - 4\left(\min\left(\left(\frac{d_P+1}{4}\right) + 2^{s-3}, 2^{s-1} - 1\right)\right) + 3 \\ &= \max\big(d_P - 2^{s-1} + 2, 2d_P - 2^{s+1} + 7\big) \\ &= \max\big(\dim(p) - 2^{s-1}, 2\dim(p) - 2^{s+1} + 3\big). \end{aligned} \tag{8.3}$$

In particular, if $\dim(p) \geq 2^s + 2^{s-1}$, then $\dim(p_{r-1}) \geq 2^s + 1$. By the separation theorem, we then have that $\mathfrak{i}_1(p_{r-1}) \leq \dim(p_{r-1}) - 2^s$, and hence

$$\mathfrak{j}_r(p) = \mathfrak{j}_{r-1}(p) + \mathfrak{i}_1(p_{r-1}) = \left(\frac{\dim(p) - \dim(p_{r-1})}{2}\right) + \mathfrak{i}_1(p_{r-1}) \leq \dim(p) - 2^s.$$

This is impossible, however. Indeed, we have $\mathfrak{j}_r(p) > d_P - a - b$ by our choice of r, and (8.2) implies that $d_P - a - b > \dim(p) - 2^s$. We must therefore have that $\dim(p) < 2^s + 2^{s-1}$. Now, since $\mathfrak{j}_r(p) > d_P - a - b$, another application of (8.2) gives that

$$\begin{aligned} \dim(p_r) &= \dim(p) - 2\mathfrak{j}_r(p) \\ &\leq \dim(p) - 2(d_P - a - b + 1) \\ &= d_P - 2(d_P - a - b) \\ &\leq d_P - 2\left(\left(\frac{d_P - 1}{2}\right) - 2^{s-2} + 1\right) \\ &= 2^{s-1} - 1. \end{aligned}$$

On the other hand, (8.3) tells us that $\dim(p_{r-1}) > 2^{s-1}$ (since $\dim(p) > 2^s$). By Karpenko's theorem on the values of the first higher Witt index ([12], [18, Prop. 10.4]), it follows that p_{r-1} is a maximal splitting form, i.e., $\dim(p_{r-1}) - \mathfrak{i}_1(p_{r-1})$ is equal to the largest power of 2 less than $\dim(p_{r-1})$. Since $\dim(p) < 2^s + 2^{s-1}$, the inequality $\dim(p_r) < 2^{s-1}$ forces that $2^{s-1} < \dim(p_{r-1}) \leq 2^s$, and so $\dim(p_{r-1}) - \mathfrak{i}_1(p_{r-1}) = 2^{s-1}$. By a result of Hoffmann ([6, Cor. 3]), there then exists an extension L/F_{r-1} over which p_{r-1} remains anisotropic, but becomes a neighbour of an s-fold Pfister form π (here we are using that $\mathrm{char}(F) \neq 2$). Let ψ denote the complementary form subform of π, and let Z be the projective quadric over L defined by its vanishing (assuming that ψ has dimension ≥ 1). By a theorem of Rost ([19, Prop. 4] or [15, Thm. 7.1]), we then have that

$$M(X'_L) \cong \left(\bigoplus_{l=0}^{\dim(p_{r-1})-2^{s-1}-1} R_\pi\{l\}\right) \bigoplus M(Z)\{\dim(p_{r-1}) - 2^{s-1}\} \tag{8.4}$$

in *Chow*(L), R_π being the Rost motive associated to π. Consider the correspondence $\theta \in \mathrm{CH}_{2^{s-1}+\mathfrak{j}_{r-1}(p)-2}(Z \times X_L)$ giving the inclusion of $M(Z)\{\mathfrak{j}_{r-1}(p)+\dim(p_{r-1})-2^{s-1}\}$ as a direct summand of $M(X_L)$. We claim that $\eta_L = \theta_*(\gamma)$, where γ is the mod-2 reduction of a torsion element of $\mathrm{CH}^{2^{s-1}+\mathfrak{j}_{r-1}(p)+a-d_P-2}(Z)$ (and θ now denotes the mod-2 reduction of the integral correspondence considered above). Before proving this, let us show how it completes the proof by way of Lemma 8.6: Observe first that

$$2^{s-1} + \mathfrak{j}_{r-1}(p) + a - d_P - 2 \leq \frac{\dim(Z)}{4}.$$

Indeed, suppose that this is not the case. Since $\dim(Z) = 2^s - \dim(p_{r-1}) - 2$, we then have that

$$4(2^{s-1} + \mathfrak{j}_{r-1}(p) + a - d_P - 2) > 2^s - \dim(p_{r-1}) - 2$$

Rearranging this inequality $\big($and using that $\dim(p_{r-1}) = \dim(p) - 2\mathfrak{j}_{r-1}(p)\big)$, we get

$$3d_P - 2^s - 2\mathfrak{j}_{r-1}(p) + 4 < 4a.$$

Together with our standing assumption on a, this yields

$$\begin{aligned} 3d_P - 2^s - 2\mathfrak{j}_{r-1}(p) + 4 &< 4\left(\left(\frac{d_P+1}{4}\right) + 2^{s-3}\right) \\ &= d_P + 2^{s-1} + 1, \end{aligned}$$

which simplifies to $2\mathfrak{j}_{r-1}(p) \geq 2\dim(p) - (2^s + 2^{s-1})$. But we then have

$$\begin{aligned} \dim(p_{r-1}) &= \dim(p) - 2\mathfrak{j}_{r-1}(p) \\ &\leq 2^s + 2^{s-1} - \dim(p) \\ &< 2^s + 2^{s-1} - 2^s \\ &= 2^{s-1}, \end{aligned}$$

contrary to the established inequality $\dim(p_{r-1}) \geq 2^{s-1}+1$. This proves the claim on the codimension of γ. At the same time, the anisotropy of $(p_{r-1})_L$ gives that

$$\begin{aligned} \mathfrak{i}_W(p_L) = \mathfrak{i}_W(p_{F_{r-1}}) &= \mathfrak{j}_{r-1}(p) \\ &\leq d_P - a - b \\ &= (2^{s-1} + \mathfrak{j}_{r-1}(p) - 2) - (2^{s-1} + \mathfrak{j}_{r-1}(p) + a - d_P - 2) - b \end{aligned}$$

and so the conditions of Lemma 8.6 are satisfied for the following data (with L, γ and θ as given):

$$Y = Z;\ \ \varphi = p_L;\ \ X = P_L;\ \ i = 2^{s-1} + \mathfrak{j}_{r-1}(p) + a - d_P - 2;$$

$$d = 2^{s-1} + \mathfrak{j}_{r-1}(p) - 2;\ \ \sigma = \eta_L.$$

It then follows that $\deg_{r-a}(\mu) = \deg_{r-a}(\mu_L) \equiv 0 \pmod 4$, as desired. It remains to prove our claim that η_L is induced from the mod-2 reduction of a torsion element of

$\mathrm{CH}^{2^{s-1}+\mathfrak{j}_{r-1}(p)+a-d_P-2}(Z)$. To this end, we consider the direct summand

$$N := \bigoplus_{l=0}^{\dim(p_{r-1})-2^{s-1}-1} R_\pi\{l\}$$

of $M(X'_L)$ from (8.4). We must again treat two cases separately.

Subcase 1. $\mathfrak{j}_{r-1}(p) \geq a - 2^{s-2} + 1$. In this case, we have

$$\begin{aligned} \dim(X') - \big(a - \mathfrak{j}_{r-1}(p)\big) &\geq \dim(X') - (2^{s-2} - 1) \\ &= \dim(p_{r-1}) - 2^{s-2} - 1 \\ &= (2^{s-2} - 1) + (\dim(p_{r-1}) - 2^{s-1}). \end{aligned}$$

Now, by another result of Rost, the group $\mathrm{CH}_t(R_\pi)$ is torsion free for all $t \geq 2^{s-2}$ (see [19, Thm. 5] or [15, Cor. 8.2]; here we are again using that $\mathrm{char}(F) \neq 2$). Setting $u := \dim(X') - \big(a - \mathfrak{j}_{r-1}(p)\big)$, it follows that the group $\mathrm{CH}_u(N)$ is torsion free. But then (8.4) implies that the pushforward θ_* identifies the torsion subgroups of $\mathrm{CH}_{u-(\dim(p_{r-1})-2^{s-1})}(Z) = \mathrm{CH}^{2^{s-1}+\mathfrak{j}_{r-1}(p)+a-d_P-2}(Z)$ and $\mathrm{CH}^a(P_L)$. This proves the claim in this case.

Subcase 2. $\mathfrak{j}_{r-1}(p) \leq a - 2^{s-2}$. Let $0 \leq m \leq 2^{s-1}$ be such that $\dim(p_{r-1}) = 2^{s-1} + m$. In this case, we then have that

$$\begin{aligned} m = \dim(p_{r-1}) - 2^{s-1} &= \dim(p) - 2\mathfrak{j}_{r-1}(p) - 2^{s-1} \\ &\geq \dim(p) - 2(a - 2^{s-2}) - 2^{s-1} \\ &= \dim(p) - 2a \\ &\geq \dim(p) - 2\left(\left(\frac{d_P+1}{4}\right) + 2^{s-3}\right) \\ &= d_P + 2 - \left(\frac{d_P+1}{2}\right) - 2^{s-2} \\ &= \left(\frac{d_P+3}{2}\right) - 2^{s-2} \\ &\geq \left(\frac{2^s+2}{2}\right) - 2^{s-2} \\ &= 2^{s-2} + 1. \end{aligned}$$

Using the inequality $a \leq 2^{s-1} - 1$, we deduce that

$$\begin{aligned} \dim(p) = 2^{s-1} + m + 2\mathfrak{j}_{r-1}(p) &\leq 2^{s-1} + m + 2(a - 2^{s-2}) \\ &= m + 2a \\ &< 2^s + m \\ &= \min(2^{s-1} + 3m, 2^s + m) \end{aligned}$$

(the last equality being valid since $m > 2^{s-2}$). Since $\dim(p_r) < 2^{s-1}$, it then follows from Corollary 5.3 that the direct summand N of $M(X'_L)$ descends to a direct summand $\widetilde{N}$ of $M(X)$. Now, because $\dim(p) > 2^s > \dim(p_{r-1})$, the upper motive $U(P)$ is a direct summand of the complementary direct summand of $\widetilde{N}$. By (8.4) (and (3.1) applied to X_L), it follows that $U(P)_L$ is isomorphic to a direct sum of Tate motives and a direct summand of $M(Z)\{\mathfrak{j}_{r-1}(p) + \dim(p_{r-1}) - 2^{s-1}\}$. Since $\eta_L \in \mathrm{CH}^a(U(P)_L)$, this again implies that $\eta_L = \theta_*(\gamma)$, where where γ is the mod-2 reduction of a torsion element of $\mathrm{CH}^{2^{s-1}+\mathfrak{j}_{r-1}(p)+a-d_P-2}(Z)$, as desired.

We have proven the proposition in the case where $K = F$. To treat the case where $K \neq F$, we simply apply the very same arguments to the form p_K (which is anisotropic). Indeed, $\deg_{d_P-a-b}(\mu)$ does not change under scalar extension, and the only issue that needs to be addressed concerns the splitting of the motivic summand $\widetilde{N}$ of $M(P)$ in subcase 2 above (using Corollary 5.3): When applying the same considerations to $M(P_K)$, we need the resulting summand to be complementary to $U(P)_K$ $\big($which could be larger than $U(P_K)\big)$. This is ensured, however, by our assumption that p_K has the same higher Witt indices as p: Indeed, under this assumption, Corollary 5.3 can be applied to P, showing that the relevant summand of $M(P_K)$ descends to a summand of $M(P)$ $\big($which is complementary to $U(P)\big)$. With this remark, the proof of the proposition is complete. □

Remark 8.8. Note that we only used the characteristic assumption on F to invoke [6, Cor. 3] and [19, Thm. 5], and that these were only used in the situation where $\dim(p) < 2^{s-1} + 2^{s-2}$ and $2d_P - 4a + 3 \leq \dim(p_{r-1}) \leq 2^s$ (r being as in the proof). In particular, the characteristic assumption is not needed if $\dim(p) \geq 2^s + 2^{s-1}$ or $4a < 2d_P - 2^s + 3$.

With the preceding proposition, we are finally ready to prove Theorem 1.2, restated here for the reader's convenience.

Theorem 8.9. *If* $\mathrm{char}(F) \neq 2$, *then Conjecture 1.1 holds in the case where* $\dim(p) > 2k - 2^{s-1}$. *In other words, if* $\mathrm{char}(F) \neq 2$ *and* $\dim(p) > 2k - 2^{s-1}$, *then* $\dim(q) = a2^{s+1} + \epsilon$ *for some integer* $a \geq 0$ *and integer* $-k \leq \epsilon \leq k$.

Proof. Recall that we are assuming that $k \leq 2^s - 2$ and that $\dim(q) > k$. Let $F = F_0 \subset F_1 \subset \cdots \subset F_{h(q)}$ be the Knebusch splitting tower of q, and let $1 \leq r \leq h(q)$ be the unique integer for which $q_r = (q_{F_r})_{\mathrm{an}}$ has dimension k. As discussed in the proof of Proposition 6.1, we then have $\dim(q_{r-1}) = 2^{n+1} - k$ for some $n \geq s$. If $n = s$, then p_{F_r} is isotropic $\big($Remark 6.2 (1)$\big)$, and the desired assertion follows from Proposition 6.1. Assume now that $n > s$, and let $K = F_r = F_{r-1}(q_{r-1})$. Since $\dim(q_{r-1}) = 2^{n+1} - k > 2^{s+1}$, the separation theorem implies that p_K is anisotropic and has the same higher Witt indices as p. Consider now the geometric correspondence $\alpha \in \mathrm{Ch}(\overline{P \times Q})$ from Proposition 8.2 and the discussion that precedes it. We claim that α does not involve the element $l_{\mathfrak{i}_1(p)-1} \times h^{(d_Q+k)/2-d_P+\mathfrak{i}_1(p)-1}$. Suppose otherwise. Since $\dim(p) > 2^s$, it follows from the separation theorem that $\mathfrak{i}_1(p) \leq \dim(p) - 2^s$. Using that $k \leq 2^s - 2$, we get that

$$\begin{aligned}
(d_Q+k)/2 - d_P + \mathfrak{i}_1(p) - 1 &\leq (d_Q+k)/2 - d_P + (\dim(p) - 2^s) - 1 \\
&= (d_Q+k)/2 - 2^s + 1 \\
&< (d_Q+k)/2 - k \\
&= (d_Q-k)/2 = \mathfrak{i}_W(q_{F(p)}).
\end{aligned}$$

Thus, multiplying α by a suitable element of the form $h^? \times h^{\mathfrak{i}_1(p)-1}$, we see that there is an F-rational element of $\mathrm{Ch}(\overline{P \times Q})$ involving $l_0 \times h^{\mathfrak{i}_W(q_{F(p)})-1}$. Since $\mathfrak{i}_W(q_K) = \mathfrak{i}_W(q_{F(p)})$, it then follows from Lemma 3.3 that p_K is isotropic, a contradiction. The claim follows, and since $\dim(p) > 2k - 2^{s-1}$, it now follows from Proposition 8.2 that $\alpha = h^0 \times l_{(d_Q-k)/2}$. Let $m = \frac{d_Q-k}{2}$. Note that since $k \leq 2^s - 2 \leq \dim(p) - 3$, we have that $n_Q - n_P < m \leq n_Q$. Since $h^0 \times l_m \in \mathrm{Ch}_{d_P+m}(\overline{P \times Q})$ is F-rational, we are in a position to attempt to apply

Proposition 7.1. To this end, let us fix an integer $\binom{\frac{d_Q+k}{2}}{} - 2^s + 2 \leq j \leq \frac{d_Q-k}{2}$. Note that

$$j > \left(\frac{d_Q+k}{2}\right) - 2^{s+1} + 2 = d_Q - \left(\frac{d_Q-k}{2}\right) - 2^{s+1} + 2 = d_Q - m - 2^{s+1} + 2,$$

i.e., j satisfies the assumption of Proposition 7.1 (with $m = \frac{d_Q-k}{2}$). Now, by our choice of K, we have that

(1) $\mathfrak{i}_W(q_K) = m+1$, and
(2) p_K is anisotropic (and so $\mathfrak{i}_W(p_K) \leq d_P - d_Q + j + m + 1$).

We claim that

(3) If $a \leq n_Q - m$ and $\eta \in \mathrm{Ch}^a(P_K)$ is the mod-2 reduction of a torsion element of $\mathrm{CH}^a(U(P)_K)$, then $\deg_{d_P-a-b}(\mu) \equiv 0 \pmod 4$ for any pair (b,μ) consisting of an integer $b \geq 0$ and an integral representative $\mu \in \mathrm{CH}^{a+b}(P_K)$ of $S^b(\eta) \in \mathrm{Ch}^{a+b}(P_K)$.

This follows from Proposition 8.7: Indeed, since p_K has the same higher Witt indices as p, the latter tells us that it will be enough to check that

$$n_Q - m \leq \min\left(\left(\frac{d_P+1}{4}\right) + 2^{s-3}, 2^{s-1} - 1\right).$$

But

$$n_Q - m = n_Q - \left(\frac{d_Q-k}{2}\right) = [k/2],$$

and a quick calculation now shows that the needed inequality follows from our assumptions that $\dim(p) > 2k - 2^{s-1}$ and $k \leq 2^s - 2$. Now, since (1), (2) and (3) hold, Proposition 7.1 tells us that the binomial coefficient

$$\binom{d_Q - m + 1}{j} = \binom{(\dim(q)+k)/2}{j}$$

is even. Since this holds for any $\left(\frac{d_Q+k}{2}\right) - 2^s + 2 \leq j \leq \frac{d_Q-k}{2}$, Lemma 8.1 then tells us that the statement of Conjecture 1.1 holds for the pair (p,q). □

Remark 8.10. The characteristic assumption was only used to verify condition (3) above (for the given K). In view of Remark 8.8, this condition is satisfied in any characteristic if $\dim(p) \geq 2^s + 2^{s-1}$ or $4(n_Q - m) < 2d_P - 2^s + 3$. Unravelling the second condition, we find that the characteristic assumption is not needed if $\dim(p) \geq 2^s + 2^{s-1}$ or $k \leq 2^{s-1}$ (note that the condition $\dim(p) > 2k - 2^{s-1}$ is vacuous in these cases).

9. A refinement for non-Pfister neighbours

In this last section, we point out that the above arguments allow for a refinement of Theorem 1.2 in the case where $k < 2^{s-1} + 2^{s-2}$. We continue with the set-up and notation of the three preceding sections. Our result is the following:

Theorem 9.1. *Suppose, in the above situation, that* $\mathrm{char}(F) \neq 2$ *and that* $k < 2^{s-1}+2^{s-2}$. *If the upper motive* $U(P)$ *is* not *a binary motive, then* $\dim(q) = a2^{s+2} + \epsilon$ *for some non-negative integer* a *and some* $-k \leq \epsilon \leq k$ *(i.e., the statement of Conjecture 1.1 holds for the pair* (p,q)*, but with the exponent of the* 2*-power raised from* $s+1$ *to* $s+2$*).*

Remark 9.2. When we say that an object in $\mathit{Chow}(F)$ is *binary*, we mean that it becomes isomorphic to a direct sum of exactly two Tate motives after scalar extension to $\overline{F}$.

Proof. Following the proof of Theorem 8.9, let $F = F_0 \subset F_1 \subset \cdots \subset F_{h(q)}$ be the Knebusch splitting tower of q, and let $0 \leq r \leq h(q)$ be such that $\dim(q_r) = k$. We claim that p_{F_r} is anisotropic. Granted this, the result is implicit in the proof of Theorem 8.9. Indeed, in the latter, we used the anisotropy of p_{F_r} together with the inequality $\dim(p) > 2k - 2^{s-1}$ (which always holds if $k < 2^{s-1}+2^{s-2}$) to show that the binomial coefficient $\binom{(\dim(q)+k)/2}{j}$ is even for any $\left(\frac{d_Q+k}{2}\right) - 2^s + 2 \leq j \leq \frac{d_Q-k}{2}$. The reader will observe, however, that we only needed here that $j \geq \left(\frac{d_Q+k}{2}\right) - 2^{s+1} + 2$. Thus, if p_{F_r} is anisotropic, then the proof of Theorem 8.9 shows that $\binom{(\dim(q)+k)/2}{j}$ is even for all $\left(\frac{d_Q+k}{2}\right) - 2^s + 1 \leq j \leq \frac{d_Q-k}{2}$, which is precisely what we need by Lemma 8.1. To finish the proof, let us suppose to the contrary that p_{F_r} is isotropic. By Remark 6.2 (1) and the proof of Proposition 6.1, this implies that $N := U(P)\{\mathfrak{i}_W(q_{F(p)}) - 1\}$ is isomorphic to a direct summand of $M(Q)$. We claim that N is a binary motive. Since N is a Tate twist of the non-binary motive $U(P)$, this will give the desired contradiction. Observe, however, that $\mathfrak{i}_r(q) > \mathfrak{i}_t(q)$ for all $r < t \leq h(q)$. Indeed, the proof of Theorem 8.9 shows that $\dim(q_{r-1}) = 2^{s+1} - k$. The former assertion then follows readily from the inequality $\dim(q_r) = k < 2^{s-1} + 2^{s-2}$ (compare the proof of Corollary 5.3). In view of Proposition 3.4, we deduce that N must be binary, and so we are done (note that N begins in the rth shell of $M(Q)$). $\square$

Remark 9.3. Again, the characteristic assumption is not needed if $\dim(p) \geq 2^s + 2^{s-1}$ or $k \leq 2^s$ (Remark 8.10).

Recall now (Remark 5.2) that it has been conjectured by Vishik that the upper motive $U(P)$ is binary if and only p is a Pfister neighbour (the 'if' implication being known by Rost). Thus, Theorem 9.1 should eventually be replaced by the following:

Conjecture 9.4. *Suppose, in the above situation, that $k < 2^{s-1} + 2^{s-2}$. If p is* not *a Pfister neighbour, then $\dim(q) = a2^{s+2} + \epsilon$ for some non-negative integer a and some $-k \leq \epsilon \leq k$ (i.e., the statement of Conjecture 1.1 holds for the pair (p, q), but with the exponent of the 2-power raised from $s+1$ to $s+2$).*

Remarks 9.5. (1) By [11], Vishik's conjecture is known to hold when $\mathrm{char}(F) \neq 2$ and $\dim(p) \leq 16$. By Theorem 9.1, it follows that Conjecture 9.4 is true in the latter case.
(2) Note that the inequality $k < 2^{s-1} + 2^{s-2}$ cannot be relaxed in Theorem 9.1 or Conjecture 9.4. Indeed, for any integer $s \geq 2$, Hoffmann has given examples of anisotropic non-Pfister neighbours p of dimension $2^s + 2^{s-2}$ with the property that $\dim(p_1) = \dim((p_{F(p)})_{\mathrm{an}}) = 2^{s-1} + 2^{s-2}$ (see [6, Ex. 2]). It is easy to check directly that the upper motives of the associated quadrics are not binary in these examples.

10. Appendix

In this short appendix, we record (for lack of reference) a couple of basic facts concerning the composition of Chow correspondences that are used in sections 7 and 8 above.

Lemma 10.1. *Let X and Y be smooth projective varieties over a field, and let π_X denote the canonical projection from $X \times Y$ to X. Then, for any $\alpha \in \mathrm{CH}(X \times Y)$, $\nu \in \mathrm{CH}(X \times X)$ and $\sigma \in \mathrm{CH}(Y)$, the following hold:*

(1) $(\pi_X)_*(\alpha \circ \nu) = \big((\pi_X)_*(\alpha)\big) \circ \nu$.
(2) $\big(([X] \times \sigma) \cdot \alpha\big) \circ \nu = ([X] \times \sigma) \cdot (\alpha \circ \nu)$.

In particular, if $\alpha \circ \nu = \alpha$, then $(\pi_X)_*\big(([X] \times \sigma) \cdot \alpha\big) \in \mathrm{CH}(X) \circ \nu$.

Proof. If $\alpha \circ \nu = \alpha$, then $\big(([X] \times \sigma) \cdot \alpha\big) \circ \nu = ([X] \times \sigma) \cdot \alpha$ by (2). Replacing α with $([X] \times \sigma) \cdot \alpha$ in (1) then yields the last statement. To prove (1) and (2), consider the commutative diagram

$$\begin{tikzcd} X \times Y \arrow[d, "\pi_X"'] & X \times X \times Y \arrow[l, "\pi_{13}"'] \arrow[r, "f"] \arrow[d, "\pi_{12}"] & X \times X \times X \times Y \arrow[d, "\pi_{123}"] \\ X & X \times X \arrow[l, "\pi_1"] \arrow[r, "g"'] & X \times X \times X \end{tikzcd}$$

where $g = \mathrm{id}_X \times \delta_X$ (δ_X being the diagonal embedding of X into $X \times X$), $f = g \times \mathrm{id}_Y$ and the other maps are the natural projections.

(1) Since the right square is cartesian, and since f and g are regular embeddings of the same codimension, we have $(\pi_{12})_* \circ f^* = g^* \circ (\pi_{123})_*$ (see [2, Prop. 55.3]). By the definition of the composition law for correspondences (see [2, §62]), we therefore have that

$$\begin{aligned} (\pi_X)_*(\alpha \circ \nu) &= (\pi_X)_*\Big((\pi_{13})_*\big(f^*(\nu \times \alpha)\big)\Big) \\ &= (\pi_X \circ \pi_{13})_*\big(f^*(\nu \times \alpha)\big) \\ &= (\pi_1 \circ \pi_{12})_*\big(f^*(\nu \times \alpha)\big) \\ &= (\pi_1)_*\Big((\pi_{12})_*\big(f^*(\nu \times \alpha)\big)\Big) \\ &= (\pi_1)_*\Big(g^*\big((\pi_{123})_*(\nu \times \alpha)\big)\Big) \\ &= (\pi_1)_*\Big(g^*\big(\nu \times (\pi_X)_*(\alpha)\big)\Big) \\ &= \big((\pi_X)_*(\alpha)\big) \circ \nu, \end{aligned}$$

as desired.

(2) Note that $f^*([X \times X \times X] \times \sigma) = [X \times X] \times \sigma = (\pi_{13})^*([X] \times \sigma)$. Since f^* is a ring homomorphism, it follows that

$$\begin{aligned} \big(([X] \times \sigma) \cdot \alpha\big) \circ \nu &= (\pi_{13})_*\Bigg(f^*\Big(\nu \times \big(([X] \times \sigma) \cdot \alpha\big)\Big)\Bigg) \\ &= (\pi_{13})_*\big(f^*([X \times X \times X] \times \sigma) \cdot f^*(\nu \times \alpha)\big) \\ &= (\pi_{13})_*\big((\pi_{13})^*([X] \times \sigma) \cdot f^*(\nu \times \alpha)\big) \\ &= ([X] \times \sigma) \cdot (\pi_{13})_*\big(f^*(\nu \times \alpha)\big) \\ &= ([X] \times \sigma) \cdot (\alpha \circ \nu) \end{aligned}$$

(here we have used the projection formula ([2, Prop. 56.9]) for the fourth equality). □

Lemma 10.2. *Let X and Y be smooth projective varieties over a field, let $\alpha \in \mathrm{Ch}(Y)$ and let $\theta \in \mathrm{Ch}(Y \times X)$. For any $b \geq 0$, we then have that*

$$S^b\big(\theta_*(\alpha)\big) = \sum_{i+j+k=b} \big(S^k(\theta)\big)_*\big(c_i(-T_Y) \cdot S^j(\alpha)\big),$$

where $-T_Y$ denotes the virtual normal bundle of Y, and $c_i(-T_Y)$ its ith Chern class modulo 2.

Proof. Let $f = \delta_Y \times \mathrm{id}_X \colon Y \times X \to Y \times Y \times X$, where δ_Y is the diagonal embedding of Y into $Y \times Y$, and let π_X denote the canonical projection from $Y \times X$ onto X. Using [2, Prop. 61.10, Thm. 61.9], we compute that

$$\begin{aligned} S^b\big(\theta_*(\alpha)\big) &= S^b\Big((\pi_X)_*\big(f^*(\alpha\times\theta)\big)\Big) \\ &= \sum_i (\pi_X)_*\Big(\big(c_i(-T_Y)\times[X]\big)\cdot S^{b-i}\big(f^*(\alpha\times\theta)\big)\Big) \\ &= \sum_i (\pi_X)_*\Big(\big(c_i(-T_Y)\times[X]\big)\cdot f^*\big(S^{b-i}(\alpha\times\theta)\big)\Big) \\ &= \sum_{i+j+k=b} (\pi_X)_*\Big(\big(c_i(-T_Y)\times[X]\big)\cdot f^*\big(S^j(\alpha)\times S^k(\theta)\big)\Big) \\ &= \sum_{i+j+k=b} (\pi_X)_*\Big(f^*\Big(\big(c_i(-T_Y)\cdot S^j(\alpha)\big)\times S^k(\theta)\Big)\Big) \\ &= \sum_{i+j+k=b} \big(S^k(\theta)\big)_*\big(c_i(-T_Y)\cdot S^j(\alpha)\big), \end{aligned}$$

as desired. □

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